MA/M.Sc.—Math-IVS (CC-402)

2019

Time : 3 hours

Full Marks : 80

Answer from both the Sections as directed

The figures in the right-hand margin indicate marks

Candidates are required to answer in their own words as far as practicable

(FUNCTIONAL ANALYSIS-II)

SECTION – A

1. Answer any *four* of the following : 4 × 4

(*a*) Let $\{x_n^1\}$ be a sequence in a normed space x. If (*i*) $\{x_n^1\}$ is bounded and (*ii*) $\{x_n^1(x)\}$ in a Cauchy sequence in k for each x in a subset of x whose span is dense in x. Then show that $\{x_n^1\}$ is weak* convergent in x_n^1. The converse holds if x is Banach space.

(Turn Over)

(*b*) Prove that Helly's selection principle.

(*c*) State and prove that Generalized polarization identity.

(*d*) Let X be an inner product space and F be a subspace of X and $x \in X$. Then $y \in F$ is a best approximation from F to x if and only if $x - y \perp F$ and in that case dist $(x, F) = < x, x - y >^{1/2}$. Justify it.

(*e*) Let $\{x_n\}$ be a sequence in a Hilbert space H. Then prove that $x_n \to x$ iff

$$x_n \xrightarrow{w} x \text{ and } \limsup_{n \to \infty} \| x_n \| \leq \| x \| .$$

Or

2. Answer *all* questions : 2 × 8

(*a*) Define weak* convergent.

(*b*) State Karlin's weak basis theorem.

(*c*) Prove that parallelogram law.

(*d*) Let X be an inner product space $\{u_1, u_2, \ldots u_n, \ldots\}$ be a countable orthonormal set in X and $k_1, k_2, \ldots$ belong to k. If $\Sigma k_n u_n$ converges to some x in X, then $< x, u_n > = k_n$ for each n and $\Sigma | k_n |^2 < \infty$.

(*e*) Let X be an inner product space and $f \in X^1$. Let $\{u_1, u_2, \ldots\}$ be an orthonormal set in X. Then show that $\Sigma |f(u_n)|^2 \leq \| f \|^2$.

(*f*) Let $\{X_n\}$ be a sequence in a Hilbert space H. Then show that

$$x_n \longrightarrow x \text{ iff } x_n \xrightarrow{w} x \text{ and } \limsup_{n \to \infty} \| x_n \| \leq \| x \| .$$

(*g*) Let X be a inner product space. Let $E \subset X$ and $x \in \overline{E}$. Then prove that there exists a best approximation from E to x iff $x \in E$.

(*h*) Let $\{A_n\}$ be a sequence of self-adjoint operators in $B < (H)$. If $0 \leq A_{n+1} \leq A_n$ for all n, then show that there is a positive operator A on H $A_n(x) \to A(x)$ for every $x \in H$.

SECTION – B

Answer all questions : 16 × 4

3. (*a*) Let X be a normed space (x_n) be a sequences in X. Then $\{x_n)$ is weak convergent in X iff

   (*i*) (x_n) is bounded sequence in X and

   (*ii*) there is some $x \in X$ such that $x'(x_n) \to x'(x)$ for every x in some subset of X' whose span is dense in X'.

Or

(*b*) Let X be a normed space $\{x'_1, x'_2, ..., x'_m\}$ be a linearly independent subset of X'. Then there are $x_1, ... x_m$ in X such that $x'_j(x_i) = \delta_{ij}$ for $i, j = 1, 2, ... m$ prove it.

4. (*a*) State and prove that the Gram-Schmidt ortho-normalization.

Or

(*b*) State and prove the Bessel's inequality.

5. (*a*) State and prove that the Riesz Representation theorem.

Or

(*b*) State and prove unique Hahn-Banach extension theorem.

6. (*a*) Show that let $A \in B < (H)$ be self-adjoint. Then A or $-A$ is a positive operator if and only if

$$|\langle A(x), y\rangle|^2 \leq \langle A(x), x\rangle \langle A(y), y\rangle$$

for all $x, y \in H$.

Or

(*b*) (*i*) Let H be a Hilbert space and $A \in B < (H)$. Then prove that there is a unique $B \in B <$ (H) such that for all $x, y \in H$, $< Ax, y > = < x, By >$.

(*ii*) Let H be a Hilbert space and $A \in B < (H)$. Show that A is normal if and only if $||A(x)|| = ||A^*(x)||$ for all $x \in H$.