Total Pages—5 **MA/M.Sc.—Math-IIS (204)**

2019

Time : 3 hours

Full Marks : 80

Answer from **both** the Sections as directed

The figures in the right-hand margin indicate marks

Candidates are required to answer in their own words as far as practicable

(ADVANCED COMPLEX ANALYSIS)

SECTION – A

1. Answer any *four* of the following questions : 4 × 4

(*a*) Evaluate :

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2 + a^2} dx; \ a > 0.$$

(Turn Over)

(b) Investigate the convergence of the infinite product :

$$\prod_{k=2}^{\infty}\left(1-\frac{1}{k^2}\right).$$

(c) Expand $f(z) = \log z$ in a Taylor series about $z = 1$.

(d) How many roots does the equation $z^7 - 2z^5 + 6z^3 - z + 1 = 0$ have in the disk $|z| < 1$?

(e) If $f(z)$ is an entire function and $f(0) \neq 0$, then show that $f(z) = f(0)\, e^{g(z)}.\, P(z)$. Where $g(z)$ is an entire function and $P(z)$ is a product of primary factors.

(f) Find the residue of $\dfrac{1}{(z^2+a^2)^2}$ at $z = ia$.

Or

2. Answer *all* questions from the following : 2 × 8

(a) Define the residue of a function at a point.

(b) Define simple periodic functions and double periodic functions.

(c) What is the value of $\Gamma\left(\frac{1}{2}\right)$?

(d) Define entire function with an example.

(e) Find the poles and residues of cot z.

(f) State Rouche's theorem.

(g) Define Weierstrass primary factors.

(h) State Euler's theorem for sequence of prime numbers.

SECTION – B

Answer **all** questions : 16 × 4

3. (a) Apply the calculus of residues to prove that

$$\int_0^{\infty}\frac{\sin \pi x}{x(1-x^2)}\,dx = \pi.$$

Or

(*b*) Let $f(z)$ be analytic except for isolated singularities a_j in a region Ω. Then prove that

$$\frac{1}{2\pi i}\int_{\gamma} f(z)\,dz = \sum_j n(\gamma, a_j)\,\mathrm{Res}_{z=a_j} f(z)$$

for any cycle γ which is homologous to zero in Ω and does not pass through any point a_j.

4. (*a*) State and prove Weierstrass factorization theorem.

Or

(*b*) Find the residue at the poles of $\Gamma(z)$.

5. (*a*) If the functions $f_n(z)$ are analytic and $\neq 0$ in a region Ω, and if $f_n(z)$ converges to $f(z)$, uniformly on every compact subset of Ω, then prove that $f(z)$ is either identically zero or never equal to zero in Ω.

Or

(*b*) Prove that

$$(2\pi)^{\frac{n-1}{2}}\,\Gamma(z) = n^{z-1/2}\,\Gamma\left(\frac{z}{n}\right)\Gamma\left(\frac{z+1}{n}\right)\cdots\cdots\Gamma\left(\frac{z+n-1}{n}\right).$$

6. (*a*) Obtain the expression of elliptic function in terms of sigma function.

Or

(*b*) Prove that

$$\wp(2z) = \frac{1}{4}\left\{\frac{\wp''(z)}{\wp'(z)}\right\}^2 - 2\wp(z)$$

where $\wp(z)$ is Weierstrass elliptic function.
