in terms of the beta function and hence evaluate

$$\int_0^1 x^5(1-x^3)^{10}\,dx.$$

Or

(*b*) Evaluate

$$\iiint_V (x+y+z)\,dx\,dy\,dz,$$

when the region V is bounded by $x+y+z=a$ $(a>0)$, $x=0$, $y=0$, $z=0$.

2019

Time : 3 hours

Full Marks : 80

Answer from **both** the Sections as directed

The figures in the right-hand margin indicate marks

Candidates are required to answer in their own words as far as practicable

(ADVANCED CALCULUS)

SECTION – A

1. Answer any *four* of the following : 4×4

(*a*) By using θ-δ technique, prove that

$$\lim_{(x,y)\to(1,1)} (x^2+2y)=3.$$

(*b*) Verify

$$\frac{\partial^2 u}{\partial x \partial y}=\frac{\partial^2 u}{\partial y \partial x},\ \text{if } u=x^y.$$

(*c*) If $x^x y^y z^z = C$, show that at $x = y = z$.

$$\frac{\partial^2 z}{\partial x \partial y} = -(x \log ex)^{-1}.$$

(*d*) Evaluate

$$\int_0^1 \int_0^{x^2} e^{y/x}\, dxdy.$$

(*e*) Find stationary points of the function

$$f(x, y) = 2xy - 3x^2 y - x^3 + x^3 y + xy^3$$

Or

2. Answer *all* the questions of the following : 2×8

(*a*) If $f(x, y) = \dfrac{xy(x^2 - y^2)}{x^2 + y^2}, (x, y) \neq (0, 0),$ when $(x, y) = (0, 0)$. Find the value of $f_{xy}(0,0)$

(*b*) Expand $f(x, y) = x^2 + xy + y^2$ in powers of $(x - 2)$ and $(y - 3)$.

(*c*) What do you know about Taylor's theorem ?

(*d*) If $z = f(x + ay) + \phi(x - ay)$, prove that

$$\frac{\partial^2 z}{\partial y^2} = a^2 \frac{\partial^2 z}{\partial x^2}.$$

(*e*) Explain explicit and implicit function with example.

(*f*) Prove that

$$\lim_{(x,y)\to(0,0)} \tan^{-1}\left(\frac{y}{x}\right)$$

does not exist.

(*g*) Evaluate

$$\iint xy(x^2 + y^2)\, dx\, dy$$

over the area between $y = x^2$ and $y = x$.

(*h*) Change the order of integration in

$$\int_0^{\infty} \int_0^{x} f(x, y)\, dx\, dy.$$

SECTION – B

Answer all questions : 16×4

3. (*a*) Expand $(x^2y + \sin y + e^x)$ in powers of $(x-1)$ and $(y-\pi)$ through quadratic terms and write the remainder, without computing θ.

Or

(*b*) Show that the function $F(x, y)$ where

$$F(x,y) = \frac{xy}{\sqrt{x^2+y^2}} \text{ if } (x,y) \neq (0,0)$$
$$= 0 \quad \text{if} \quad (x,y) = (0,0)$$

is continuous possesses partial derivatives but is not totally differentiable at (0, 0).

4. (*a*) If V be a function of r alone, where $r^2 = x^2$. show that

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = \frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r}.$$

Or

(*b*) If $x = r\cos\theta$, $y = r\sin\theta$, prove that

$$xy\left(\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2}\right) - (x^2 - y^2)\frac{\partial^2 u}{\partial x \partial y} = 0,$$

becomes $\dfrac{\partial^2 u}{\partial r \partial \theta} - \dfrac{\partial u}{\partial \theta} = 0$, where $u = u(r, \theta)$.

5. (*a*) If λ, μ and ν are the roots of the equation in k,

$$\frac{x}{a+k} + \frac{y}{b+k} + \frac{z}{c+k} = 1,$$

prove that

$$\frac{\partial(x,y,z)}{\partial(\lambda,\mu,\nu)} = -\frac{(\mu-\nu)(\nu-\lambda)(\lambda-\mu)}{(b-c)(c-a)(a-b)}.$$

Or

(*b*) Prove that all rectangular parallelopiped of the same volume, the cube has the least surface.

6. (*a*) Express

$$\int_0^1 x^m (1-x^n)^p \, dx$$