MA/M.Sc.—Math-IS(101)

2019

(January)

Time : 3 hours

Full Marks : 80

Answer from **both** the Sections as directed

The figures in the right-hand margin indicate marks

Candidates are required to answer in their own words as far as practicable

(PARTIAL DIFFERENTIAL EQUATION AND APPLICATION)

SECTION – A

1. Answer any *four* of the following : 4×4

(*a*) Find the general solution of

$$u_{xx} + u_{yy} = 0$$

(*b*) Determine the general solutions of the following equation :

$$u_{xx} - \frac{1}{C^2} u_{yy} = 0, \quad C = \text{constant.}$$

(Turn Over)

(*c*) Determine the solution of each of the following initial value problem :

$$u_{tt} - C^2\ u_{xx} = 0,\ u(x, 0) = x^3\ u_t(x, 0) = x$$

(*d*) Find the temperature distribution in a rod of length *l*. The faces are insulated and the initial temperature distribution is given by $x(l - x)$.

(*e*) Find the Green's function for each of the following problem $L[y] = y'' = 0$, $y(0) = 0$, $y'(1) = 0$.

(*f*) Find the Fourier transform of

$$f(x) = \exp(-ax^2).$$

Or

2. Answer *all* the questions : 2 × 8

(*a*) What is fourier sine transform ?

(*b*) Define characteristic function and eigen-value.

(*c*) What are the properties of Laplace trans-form ?

(*d*) Define Green's function.

(*e*) What is the difference between Neumann's problem and Dirichlet problem ?

(*f*) State maximum principle.

(*g*) State Schrödinger equation.

(*h*) What is linear Harmonic oscillator ?

SECTION – B

Answer **all** questions : 16 × 4

3. (*a*) (*i*) Verify that $u(x, y) = x^3 + y^2 + e^x$ $(\cos x \sin y \cosh y - \sin x \cos y \sinh y)$ is a classical solution of the Poisson equation

$$u_{xx} + u_{yy} = (6x + 2).$$

(*ii*) Show that $u(x, y\,; k) = e^{-ky}\sin(kx)$, $x \in \mathbb{R}$, $y > 0$, is a solution of the equation

$$\nabla^2 u \equiv u_{xx} + u_{yy} = 0$$

for any real parameter k.

Or

(*b*) (*i*) Obtain the general solution of the following equation :

$$x^2 u_{xx} + 2xy u_{xy} + y^2 u_{yy} + xy u_x + y^2 u_y = 0$$

(*ii*) Determine the region in which the given equation is hyperbolic, parabolic or elliptic, and transform the equation in the respective region to canonical form :

$$u_{xx} + u_{xy} - x u_{yy} = 0$$

4. (*a*) (*i*) Solve the initial value problem

$$u_{xx} + 2u_{xy} - 3u_{yy} = 0$$
$$u(x, 0) = \sin x, \ u_y(x, 0) = x.$$

(*ii*) Determine the solution of the initial boundary-value problem

$$u_{tt} = 9u_{xx}, \quad 0 < x < \infty, \ t > 0$$
$$u(x,0) = 0, \quad 0 \le x < \infty$$
$$u_t(x,0) = x^3, \quad 0 \le x < \infty$$
$$u_x(0, t) = 0, \quad t \ge 0.$$

Or

(*b*) (*i*) Find the solution of the following problem :

$$u_{tt} = C^2 u_{xx} = 0, \quad 0 < x < \pi, \ t > 0$$
$$u(x,0) = \sin x, \quad u_t(x,0) = x^2 - \pi x, \ 0 \le x \le \pi,$$
$$u(0,t) = u(\pi,t) = 0, \ t > 0$$

(*ii*) Solve the telegraph equation by the method of separation of variables

$$u_{tt} + a u_t + b u = C^2 u_{xx}, \ 0 < x < l, \ t > 0,$$
$$u(x,0) = f(x), \ u_t(x,0) = 0,$$
$$u(0,t) = u(l,t) = 0, \quad t > 0$$

5. (*a*) (*i*) Determine all eigenvalues and eigen functions of the Sturm-Liouville system

$$x^2 y'' + xy' + \lambda y = 0,$$
$$y(1) = 0, \quad y, y'$$

are bounded at $x = 0$.

(*ii*) Expand the function $f(x) = \sin x$, $0 \le x \le \pi$ in terms of the eigen functions of the Sturm-Liouville problem

$$y'' + \lambda y = 0$$
$$y(0) = 0,\ y(\pi) + y'(\pi) = 0$$

Or

(*b*) (*i*) Show that

$$\left.\frac{dG(x,\xi)}{dx}\right|_{\xi=x^-}^{\xi=x^+} = \frac{1}{p(x)}$$

is equivalent to $\left.\frac{dG(x,\xi)}{dx}\right|_{x=\xi^-}^{x=\xi^+} = -\frac{1}{p(\xi)}$.

(*ii*) Solve the Robin problem

$$\nabla^2 u = -r^2 \sin 2\theta$$
$$u(r_1, \theta) = 0,\ u_r(r_2, \theta) = 0$$

6. (*a*) (*i*) Solve $\nabla^2 u = 0,\ a < r < b,\ 0 < \theta < 2\pi$

$$u_r(a,\theta) + hu(a,\theta) = f(\theta),$$
$$u_r(b,\theta) + hu(b,\theta) = g(\theta).$$

(*ii*) Solve the following Neumann problem :

$$\nabla^2 u = 0,\ 0 < x < \pi,\ 0 < y < \pi,$$
$$u_x(0, y) = (y - \pi/2),\ u_x(\pi, y) = 0,\ 0 \le y \le \pi,$$
$$u_y(x, 0) = x,\ u_y(x, \pi) = x,\ 0 \le x \le \pi.$$

Or

(*b*) (*i*) Show that

$$\int_0^\infty e^{-a^2x^2} \cos bx\,dx = \left(\sqrt{\pi}/2a\right) e^{-b^2/4a^2}, a > 0$$

(*ii*) Prove that

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-k^2t - ikx}\, dk = \frac{1}{\sqrt{2t}}\, e^{-x^2/4t}$$