M.Sc.—Math-IIIS (AE-307)

2019

(January)

Time : 3 hours

Full Marks : 80

Answer from **both** the Sections as per direction

The figures in the right-hand margin indicate marks

Candidates are required to answer in their own words as far as practicable

(MATHEMATICAL STATISTICS - II)

SECTION – A

1. Answer any *four* questions : 4 × 4

(*a*) What is ψ^2-distribution with *n*-degress of freedom. What is its p.d.f.

(*b*) Show that the sequence of characteristic functions $\psi_n(t) = \left(\frac{\sin nt}{nt}\right)$ converges to a limit function $\psi(t)$.

(Turn Over)

(*c*) (X, Y) have joint p.d.f. $f(x, y)$ show that for the p.d.f. $f(x, y) = 6xy^2$, $0 < x, y < 1$, X, Y are stochastically independent.

(*d*) X has a uniform distribution over $(0, 1)$. Find the distribution of $\frac{1}{x}$.

(*e*) X is uniformly distributed over $(-a, a)$. Find the distribution of $Y = \cos x$.

(*f*) Show that $\rho_{1\cdot 23} \geq \rho_{12}$.

Or

2. Answer *all* questions : 2×8

(*a*) What do you mean by multivariate distribution.

(*b*) Define Gamma distribution.

(*c*) What is Gauss' multiplication formula ?

(*d*) What is Beta function ?

(*e*) Define extreme value distribution ?

(*f*) Define Weibull distribution.

(*g*) What is conditional p.d.f. ?

(*h*) What is conditional density ?

SECTION – B

Answer **all** questions : 16×4

3. (*a*) (*i*) $\overline{X}$ is mean of n independent $N(\mu, \sigma^2)$ variables. Show that for any $\in > 0$,

$$\lim_{n \to \infty} P(|\overline{x} - \mu| \geq \in) = 0.$$

(*ii*) Let N be a non-negative random integer with geometric distribution,

$$P(N = n) = (1 - e^{-\mu})e^{-n\mu}, \; n = 0, 1, 2, \ldots$$

Let M be a positive integer with p.m.f.

$$P(M = m) = \frac{e^{-\mu}\mu^m}{\underline{|m}\left(1 - e^{-\mu}\right)}, \; m = 1, 2, \ldots$$

Let $(X_1, \ldots, X_M)$ be M independent variables, each uniform over $(0, 1)$. Show that

$$Y = \mu\{N + mn(X_1, ..., X_N)\}$$

has the standard exponential distribution.

Or

(*b*) (*i*) Show that for the power function distribution

$$f(x) = Cx^{c-1},\ 0 \le x \le 1,\ \mu_r^1 = \frac{c}{c+r},$$

$$\mu_2 = \frac{c}{(c+2)(c+1)^2}$$

mode = 1 for $c > 1$, 0 for $c < 1$

median $= \left(\frac{1}{2}\right)^{\frac{1}{c}}$.

(*ii*) Show that for Erlang distribution

$$f(x) = \frac{\left(\frac{x}{b}\right)^{c-1} e^{-x/b}}{b\left[(c-1)!\right]}, 0 \le x \le \infty, b > 0, c(\text{integer}) > 0$$

m.g.f. is $(\text{hbt})^{-c}$, $t > \frac{1}{b}$, $\mu^1 = bc, \mu_2 = b^2c,$

$\sqrt{\beta_1} = \frac{2}{\sqrt{c}}$, $\beta_2 = 3 + \frac{6}{c}$, mode is $b(c-1)$.

4. (*a*) (*i*) For a given λ, X follows Poisson with parameter λ, λ follows a gamma distribution

$$f_\lambda(\lambda) = \frac{r^p}{\sqrt{(p)}} e^{-r\lambda}\lambda^{p-1}, \lambda > 0, r > 0, p(\text{integer}) > 0$$

Find the overall distribution of X, its mean and variance.

(*ii*) $f_{X,Y}(x, y) = \frac{3x+y}{4} e^{-x-y}, x > 0, y > 0.$

Find the marginal densities, conditional densities and $\rho(X, Y)$

Or

(*b*) (*i*) (X, Y) have joint p.d.f.

$$f(x, y) = \frac{3}{2\sqrt{x}}, 0 < y < x < 1.$$

Find $f_{Y|X}\left(y \mid x=\frac{1}{2}\right), f_{X|Y}\left(x \mid y=\frac{1}{2}\right)$.

(*ii*) (X, Y) follows a bivariate normal distribution with $(0, 0, \sigma_X^2, \sigma_Y^2, \rho)$. Show that

$$U = \frac{X}{\sigma_X} - \rho\frac{Y}{\sigma_Y}$$

and $V = \left(\sqrt{1-\rho^2}\; Y / \sigma_Y\right)$ are independent variates.

5. (*a*) (*i*) Let $\{X_n\}$ be a sequence of independent r.v.'s and p.m.f. of X_n is

$$P(X_n = -n-4) = \frac{1}{n+4},$$

$$P(X_n = -1) = 1 - \frac{4}{n+4},$$

$$P(X_n = n+4) = \frac{3}{n+4},$$

show that $p \lim_{n\to\infty} X_n = -1$.

(*ii*) Let $\{F_n\}$ be a sequence of d.f.'s such that

$$F_n(x) = \begin{cases} 0, & x < \theta + \frac{1}{n} \\ 1, & x \ge \theta + \frac{1}{n} \end{cases}$$

Show that the corresponding sequence of random variables $X_n \xrightarrow{L} X$ where X is a r.v. degenerate at $X = \theta$.

Or

(*b*) (*i*) Let $F_n(x)$ be a sequence of negative binomial distributions with parameters $(r_n . q_n)$ with $r_n \to \infty$, $q_n \to 0$ in such a way that $r_n q_n \to \lambda$. Show that $F_n(x)$ converges in distribution to Poisson distribution.

(*ii*) Let $g(y)$ be a continuous function and let $\{X_n\}$ be a sequence of r.v.'s such that $X_n \xrightarrow{P} X$. Show that $g(X_n) \xrightarrow{P} g(X)$.

6. (*a*) (*i*) Prove that the necessary and sufficient conditions for the coincidence of three regression planes for a bivariate distribution is

$$\rho_{12}^2+\rho_{13}^2+\rho_{23}^2-2\rho_{12}\,\rho_{13}\,\rho_{23}=1.$$

(*ii*) If the random vector $X=(X_1, \ldots, X_4)'$ have covariance matrix :

$$\Sigma=\begin{bmatrix} \sigma^2 & \sigma_{12} & \sigma_{13} & \sigma_{14} \\ \sigma_{12} & \sigma^2 & \sigma_{14} & \sigma_{13} \\ \sigma_{13} & \sigma_{14} & \sigma^2 & \sigma_{12} \\ \sigma_{14} & \sigma_{13} & \sigma_{12} & \sigma^2 \end{bmatrix}$$

show that the four multiple correlation coefficients between one variable and the other three are equal.

Or

(*b*) (*i*) Show that if in a *p*-variate distribution

all the correlations $\rho_{ij}(i \neq j)$ are equal to ρ,

$$\rho \geq -\frac{1}{p-1}.$$

(*ii*) Show that if $\rho_{1.\,2,\,3,\,\ldots\,p}$ is zero, all partial correlation coefficients of all orders vanish.