MA/M.Sc.—Math-IS(104)

2019

(January)

Time : 3 hours

Full Marks : 80

Answer from **both** the Sections as directed

The figures in the right-hand margin indicate marks

Candidates are required to answer in their own words as far as practicable

(ELEMENTARY COMPLEX ANALYSIS)

SECTION – A

1. Attempt any *four* of the following : 4×4

(*a*) Prove that

$$\left|\frac{a-b}{1-\bar{a}b}\right|<1$$

where $|a|<1$ and $|b|<1$.

(*b*) Find the center and the radius of the circle which circumscribes the triangle with the vertices a_1, a_2, a_3. Express the result in symmetric form.

(*c*) Prove that the functions $u(z)$ and $u(\bar{z})$ are simultaneously harmonic.

(*d*) Discuss the uniform convergence of the series

$$\sum_{n=1}^{\infty} \frac{x}{n(1+nx^2)}$$

for real values of x.

(*e*) Show that any linear transformation which transforms the real axis into itself can be written with real coefficients.

(*f*) Verify that the inside of the circle $|z-a| = R$ is formed by all points z with $|z-a| < R$

Or

2. Answer any *eight* questions : 2×8

(*a*) Find the linear transformation which carries $0, i, -i$ into $1, -1, 0$.

(*b*) If $T_1 z = \frac{z+2}{z+3}$, $T_2 z = \frac{z}{z+1}$,

find $T_1 T_2 z$, $T_2 T_1 z$ and $T_1^{-1} T_2 z$.

(*c*) Define a power series.

(*d*) Prove that a convergent sequence is bounded.

(*e*) If a rational function is real on $|z| = 1$, how are the zeros and poles situated ?

(*f*) Express $\cos 3\phi$ in terms of $\cos\phi$ and $\sin\phi$.

(*g*) Express the fifth and tenth roots of unity in algebraic form.

(*h*) Find the symmetric points of a with respect to the lines which bisect the angles between the coordinate axes.

(*i*) What is a pole ?

(*j*) What is residue ?

SECTION – B

Answer all questions from the following : 16×4

3. (*a*) (*i*) Show that all circles that pass through a and $\frac{1}{a}$ intersect the circle $|z| = 1$ at right angles.

(*ii*) Prove analytically that the midpoints of parallel chords to a circle lie on a diameter perpendicular to the chords.

(*iii*) When does $az + b\overline{z} + c = 0$ represent a line ?

Or

(*b*) (*i*) If $\sum_{0}^{\infty}$ an converges, then prove that

$$f(z) = \sum_{0}^{\infty} a_n z^n$$

tends to $f(1)$ as z approaches 1 in such a way that $|1 - z| / (1 - |z|)$ remains bounded.

(*ii*) If $\sum a_n z^n$ has radius of convergence R, what is the radius of convergence of $\sum a_n z^{2n}$?

4. (*a*) (*i*) If all zeros of a polynomial $P(z)$ lie in a half plane, then prove that all zeros of the derivative $P'(z)$ lie in the same half plane.

(*ii*) Expand $(1 - z)^{-m}$, m a positive integer, in powers of z.

Or

(*b*) (*i*) Show that z and z' correspond to diametrically opposite points on the Riemann sphere if and only if $z\overline{z}' = -1$.

(*ii*) Find the radius of the spherical image of the circle in the plane whose center is a and radius R.

5. (*a*) (*i*) Give a precise definition of a single -valued branch of $\sqrt{1+z} + \sqrt{1-z}$ in a suitable region, and prove that it is analytic.

(*ii*) If the consecutive vertices z_1, z_2, z_3, z_4

of a quadrilateral lie on a circle, prove that

$$|z_1 - z_3| \cdot |z_2 - z_4| = |z_1 - z_2| \cdot |z_3 - z_4| + |z_2 - z_3| \cdot |z_1 - z_4|$$

Or

(*b*) (*i*) Find the linear transformation which carries the circle $|z| = 2$ into $|z + 1| = 1$, the point-2 into the origin, and the origin into i.

(*ii*) Prove that every reflection carries circles into circles.

6. (*a*) (*i*) Compute $\int_{|z|=2} \frac{dz}{z^2 - 1}$ for the positive sense of the circle.

(*ii*) Prove that a function which is analytical in the whole plane and satisfies an inequality $|f(z)| < |z|^n$ for some n and all sufficiently large $|z|$ reduces to a polynomial.

Or

(*b*) (*i*) State and prove maximum principle.

(*ii*) Prove by use Schwarz' lemma that every one-to-one conformal mapping of a disk onto another is given by a linear transformation.