(*ii*) Prove that a polynomial of degree n over a field can have at most n roots in any extension field.

Or

(*b*) (*i*) Let $f(x) \in F[x]$ be of degree $n \geq 1$. Then prove that there is an extension E of F of degree at most $\underline{|n}$ in which $f(x)$ has n roots.

(*ii*) If F is of characteristic 0 and if a, b are algebraic over F, then prove that there exists an element $c \in F(a, b)$ such that $F(a, b) = F(c)$.

MA/M.Sc.—Math-IS (103)

2019

(January)

Time : 3 hours

Full Marks : 80

Answer from both the Sections as directed

The figures in the right-hand margin indicate marks

Candidates are required to answer in their own words as far as practicable

(ALGEBRA)

SECTION – A

1. Answer any *four* of the following : 4 × 4

(*a*) Let G be a group, H a subgroup of G, T an automorphism of G. Let $(H)T = \{hT \mid h \in H\}$. Prove $(H)T$ is a subgroup of G.

(*Turn Over*)

(*b*) If H is a subgroup of G, show that for every $g \in G, gHg^{-1}$ is a subgroup of G.

(*c*) Express the following as the product of disjoint cycles :

$$(1, 2, 3)(4, 5)(1, 6, 7, 8, 9)(1, 5)$$

(*d*) Compute $a^{-1}ba$, where

$$a = (1, 3, 5)(1, 2), b = (1, 5, 7, 9)$$

(*e*) List all the conjugate classes in S_3 and verify the class equation.

(*f*) Prove that if $[a, b] = [a', b']$ and $[c, d] = [c', d']$ then $[a, b][c, d] = [a', b'][c', d']$.

(*g*) In a commutative ring with unit element, prove that the relation a is an associate of b is an equivalence relation.

(*h*) Find the greatest common divisor in $J[i]$ of $3 + 4i$ and $4 - 3i$.

Or

2. Answer *all* the questions : 2 × 8

(*a*) Find the orbits and cycles of the following permutations :

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 3 & 4 & 5 & 1 & 6 & 7 & 9 & 8 \end{pmatrix}.$$

(*b*) Prove that $(1, 2, \ldots, n)^{-1} = (n, n-1, \ldots, 2, 1)$.

(*c*) State Cayley theorem.

(*d*) G is a cyclic group of order 12, $T : x \to x^3$, is T is a automorphism of G ?

(*e*) Give an example of a dihedral group of order $2n$.

(*f*) Define p-Sylow subgroup.

(*g*) Define Euclidean ring.

(*h*) State Remainder theorem.

SECTION – B

Answer **all** questions of the following : 16×4

3. (*a*) (*i*) If G is a group, then prove that $\mathscr{A}(G)$, the set of automorphisms of G, is also a group.

   (*ii*) If p is a prime number and $p \mid O(G)$, then prove that G has an element of order p.

Or

   (*b*) (*i*) If $O(G) = p^n$ where p is a prime number, then prove that $\mathbb{Z}(G) \neq (e)$.

   (*ii*) Prove Sylow theorem.

4. (*a*) (*i*) Let R be a commutative ring with unit element whose only ideals are (O) and R itself. Then prove that R is a field.

   (*ii*) Let R be a Euclidean ring and $a, b \in R$. If $b \neq 0$ is not a unit in R, then prove that $d(a) < d(ab)$.

Or

   (*b*) (*i*) State and prove unique factorization theorem.

   (*ii*) Find the g.c.d. in $J[i]$ of $3 + 4i$ and $4 - 3i$.

5. (*a*) (*i*) If V is the internal direct sum of $U_1, \ldots, U_n$ then prove that V is isomorphic to the external direct sum of $U_1, \ldots, U_n$.

   (*ii*) Prove that $L(S)$ is a subspace of V.

Or

   (*b*) (*i*) If $v_1, \ldots, v_n$ are in V, then prove that either they are linearly independent or some v_k is a linear combination of the preceding ones, $v_1, \ldots, v_{k-1}$.

   (*ii*) If $v_1, \ldots, v_n$ is a basis of V over F and if $w_1, \ldots, w_n$ in V are linearly independent over F, then prove that $m \leq n$.

6. (*a*) (*i*) Prove that the element $a \in K$ is algebraic over F if and only if $F(a)$ is a finite extension of F.