Or

(*b*) Let F be a subspace of X and $x \in X$. Then $y \in F$ is a best approximation from F to x if and only if $x - y \perp F$ and in that case dist $(x, F) = \langle x, x-y \rangle^{1/2}$.

6. (*a*) Let $A \in BL(H)$ be self-adjoint. Then A or $-A$ is a positive operator if and only if

$$|\langle A(x), y\rangle|^2 \le \langle A(x), x\rangle \langle A(y), y\rangle$$

for all $x, y \in H$.

Or

(*b*) Let H_1 and H_2 be Hilbert spaces and $A : H_1 \to H_2$ be linear. A is continuous iff the set $g(A) = \{(x, A(x)) : x \in H_1\}$ is closed in the Hilbert space $H_1 \times H_2$ with the inner product

$$\langle (x_1, x_2), (y_1, y_2)\rangle = \langle x_1, y_1\rangle + \langle x_2, y_2\rangle.$$

2018

Time : 3 hours

Full Marks : 80

Answer from **both** the Sections as directed

The figures in the right-hand margin indicate marks

Candidates are required to answer in their own words as far as practicable

(FUNCTIONAL ANALYSIS-II)

SECTION – A

1. Answer any *four* of the following : 4 × 4

(*a*) Let X be a separable normed space. Then prove that every bounded sequence in X' has a weak* convergent subsequence.

(*b*) Let X be a reflexive normed space. Then show that every closed subspace of X is reflexive.

(*c*) Let $\{u_1, u_2, \cdots\}$ be a countable orthonormal set in an inner product space x and $x \in X$ then prove that

$$\sum |\langle x, u_n \rangle|^2 \le \|x\|^2,$$

where equality holds iff

$$x = \sum_n \langle x, u_n \rangle u_n.$$

(*d*) Let X be an inner product space. If $E \subset X$ is convex, then their exists at most one best approximation from E to any $x \in X$. Prove it.

(*e*) Let H be a Hilbert space and $A \in BL(H)$ then show that

$$\|A^*\| = \|A\| \text{ and } \|A^* A\| = \|A^2\| = \|AA^*\|.$$

(*f*) Let (x_n) be a sequence in a Hilbert space H. Then prove that $x_n \to x$ iff $x_n \xrightarrow{w} x$.

Or

2. Answer *all* questions : 2 × 8

(*a*) Let X be a reflexive normed space. Then show that X' is separable if X is separable.

(*b*) Define weak convergent.

(*c*) If X is a Hilbert space and

$$\sum_n |k|^2 < \infty,$$

then show that

$$\sum_n k_n u_n$$

converges in X, where X is an inner product space and $\{u_1, u_2, \cdots\}$ is a countable orthonormal set in X.

(*d*) Let X be a inner product space, every finite dimensional subspace of x is closed in X.

(*e*) Let $\{x_n\}$ be a sequence in a Hilbert space H. If $\{x_n\}$ is bounded, then prove that it has a weak convergent subsequences.

(*f*) State Riesz representation theorem.

(*g*) Let H be a Hilbert space and $A \in BL(H)$. Show that the closure of $R(A)$ equals $Z(A^*)^{\perp}$.

(*h*) Let $x \in X$. Then prove that $\langle x, y \rangle = 0$ for all $y \in X$ if $x = 0$

SECTION – B

Answer **all** questions : 16 × 4

3. (*a*) Let $\{z_n\}$ be a sequence of non-decreasing functions on $[a, b]$ such that $\alpha \le z_n(t) \le \beta$ for some constants α, β all $n = 1, 2, \cdots$ and $t \in [a, b]$. Then there is a non-decreasing function z on $[a, b]$ such that z is right continuous on (a, b) and for some subsequence $\{z_{n_j}\}$ of $\{z_n\}$, we have $z_{n_j}(a) \to z(a)$, $z_{n_j}(b) \to z(b)$ and $z_{n_j}(t) \to z(t)$ for every $t \in (a, b)$ at which z is continuous.

Or

(*b*) Let X and Y be normed space and $F : X \to Y$ be a linear map. Consider the following conditions :

(*i*) whenever $x_n \to x$ in X, we have $F(x_n) \to F(x)$ in Y

(*ii*) whenever $x_n \xrightarrow{w} x$ in X, we have $F(x_n) \xrightarrow{w} F(x)$ in Y

(*iii*) whenever $x_n \to x$ in X, we have $F(x_n) \xrightarrow{w} F(x)$ in Y

(*iv*) whenever $x_n \xrightarrow{w} x$ in X, we have $F(x_n) \to F(x)$ in Y

Show that the conditions (*i*), (*ii*) and (*iii*) are equivalent. Condition (*iv*) implies condition (*ii*), but not converse.

4. (*a*) State and prove that the Coram-Schmidt orthonormalization theorem.

Or

(*b*) Let H be a nonzero Hilbert space over K. Then the following conditions are equivalent :

(*i*) H has a countable orthonormal basis.

(*ii*) H is linearly isometric to K^n for some n.

(*iii*) H is separable.

5. (*a*) Let E be a non-empty closed convex subset of a Hilbert space H. Then for each $x \in H$, there exists a unique best approximation from E to x.