(*b*) Show that a discrete module consists either of zero alone, of the integral multiples η_w of a single complex number $w = 0$, or of all linear combinations $n_1w_1 + n_2w_2$ with integral cofficients of two numbers w_1, w_2 with non-real ratio w_2/w_1 ? 8

Or

(*c*) Prove that every point τ in the upper half plane in equivalent under the congruence subgroup mod 2 to exactly one point in $\bar{\Omega} \cup \Omega'$? 8

(*d*) Prove that

$$\varphi(2z) = \frac{1}{4}\left(\frac{\varphi''(z)}{\varphi'(z)}\right)^2 - 2\varphi(z).$$ 8

MA/M.Sc.—Math-IIS(204)

2018

Time : 3 hours

Full Marks : 80

Answer from **both** the Sections as per direction

The figures in the right-hand margin indicate marks

Candidates are required to answer in their own words as far as practicable

(ADVANCED COMPLEX ANALYSIS)

SECTION – A

1. Answer *all* questions : 2×8

(*a*) State Rouche's theorem.

(*b*) Find an expressions for the residue of $f(z)$ at a simple pole $z = a$.

(*c*) Define Weistrass's theorem.

(*d*) Define Gamma function.

(*e*) Define Riemann zeta function.

(*f*) Write Jensons's formula.

(*g*) What do you mean period module ?

(*h*) Define elliptic modular function ?

Or

2. Answer *all* questions : 4×4

(*a*) Find the residue of $\frac{z^2}{z^2+a^2}$ at $z = ai$.

(*b*) Explain Taylor series to an analytic function.

(*c*) Explain Hardmard's theorem.

(*d*) Show that an elliptic function without poles is a constant.

SECTION – B

3. (*a*) Let $f(z)$ be analytic except for isolated singularities a_j in a region Ω then proof that

$$\frac{1}{2\pi i}\int_r f(z)\,dz = \sum_j n(r, a_j) \operatorname{Res}_z = a_j\ f(z).$$ 8

(*b*) Evaluate the integral by residue method.

$$\int_0^\infty \frac{\cos x}{x^2+a^2}\,dx,\ a \text{ is real.}$$ 8

Or

(*c*) Find the residues at the pole of the function

$$f(z)\frac{z^4}{(c^2+z^2)^4}$$ 8

(*d*) Evaluate the integral by residue method

$$\int_0^\infty \frac{\log x}{1+x^2}\,dx$$ 8

4. (*a*) If the functions $f_n(z)$ are analytic and $\neq 0$ in a region Ω and if $f_n(z)$ converges to $f(z)$, uniformly on every compact subset of Ω, then show that $f(z)$ is either identically zero or never equal to zero in Ω. 8

(b) Express

$$\sum_{-\infty}^{\infty} \frac{1}{z^3 - n^3}$$

in closed form. 8

Or

(c) Proof that

$$\sum_{n=1}^{\infty} \frac{nz^n}{1-z^n} = \sum_{n=1}^{\infty} \frac{z^n}{(1-z^n)^2}$$

For $|z| < 1$. 8

(d) Show that the infinite product $\prod_1^{\infty} (1+a_n)$ with $1 + a_n \neq 0$ converges simultaneously with the series $\sum_1^{\infty} \log(1+a_n)$ whose terms represent the values of the principal branch of the logarithm ? 8

5. (a) Prove that the genus and the order of an entire functions satisfy the double inequality $h \leq \lambda \leq h+1$? 8

(b) Show that the function

$$\xi(s) = \frac{1}{2} s(1-s) \pi - s/2 \ \Gamma(s/2) \xi(s)$$

in entire and satisfies $\xi(s) = \xi(1-s)$? 8

Or

(c) Show that an entire function of fractional order assumes every finite value infinitely many times. 8

(d) Prove that for $\sigma > 1$

$$\xi(s) = -\frac{\Gamma(1-s)}{2\pi i} \int_c \frac{(-z)^{s-1}}{e^z - 1} dz$$

where $(-z)^{s-1}$ is defined on the complement of the positive real axis as

$e^{(s-1)} \log(-z)$ with $-\pi < \operatorname{Im} \log(-z) < \pi$ 8

6. (a) Prove that the zeros $a_1, \ldots\ldots, a_n$ and poles $b_1, \ldots, b_n$ of an elliptic function satisfy

$$a_1 + \ldots a_n \equiv b_1 + \ldots + b_n \pmod{M}$$ 8