zero as $t \to \infty$ if the condition $||f(t, y)|| \le \alpha(t)\, ||y||$, $t \in J$ where $\alpha : J \to R^t$ is continuous on J and such that $\int_0^\infty \alpha(t)dt$ is finite and that $\alpha(t) \to 0$ as $t \to \infty$ holds.

10. Prove that the null solution of the equation $x' = A(t)x$:

(a) Is stable if and only if there exists a positive constant K such that $||\Phi(t)|| \le K$, $t \ge t_0$.

(b) Is asymptotically stable if and only if $||\Phi(t)|| \to 0$ as $t \to \infty$.





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# 2013

*Time : 4 hours*

*Full Marks : 100*

*The questions are of equal value.*

*Answer any **five** questions.*

## (ORDINARY DIFFERENTIAL EQUATIONS)

1. (a) Prove that if a(t) and b(t) are continuous functions on an interval I then there exists a solution x(t) of the equation $x' + a(t)\,x = b(t)$, $t \in I$ on I passing through $(t_0, x_0)$ and $x(t) = \exp$

$$\left(-\int_{t_0}^{t} a(s)\,ds\right) x_0 + \int_{t_0}^{t} \exp\left(-\int_{s}^{t} a(u)\,du\right) b(s)\,ds.$$

(b) Solve :

(i) $x\,dt + (t - x^3)\,dx = 0$

(ii) $x^2\,dt = (x\,dt - t\,dx)$

2. (a) Find the general solutions of :

(i) $x''' + 6x'' + 11x' + 6x = 0$

(ii) $x^{(4)} - 16x = 0$

(b) Prove that if x(t) is the solution of $L(x) = x'' + b_1(t)x' + b_2(t)x = h(t)$, $t \in I$ then :

$$x(t) = \int_{t_0}^{t} \frac{[x_1(s)\,x_2(t) - x_2(s)x_1(t)]\,h(s)}{w(x_1, x_2)(x)}\,dx$$

where $x_1(t)$ and $x_2(t)$ are two linearly independent solutions of $L(y) = 0$.

3. (a) Prove that the set of all solutions of the system $x' = A(t)x$ where A(t) is a continuous $n \times n$ matrix in I forms an n-dimensional vector space over the field of complex numbers.

(b) Let $\Phi$ be a fundamental matrix for the system $x' = A(t)x$ where A(t) is a continuous $n \times n$ matrix and let C be a constant non-singular matrix. Prove that $\Phi$ C is also a fundamental matrix for the system and every fundamental matrix of the system is of this type for some non-singular matrix C.



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on $[t_1, t_2]$. Moreover, in this case the conclusion is still true if the solution y(t) is linearly independent of x(t).

(b) Prove that the Euler's equation $x'' + \frac{k}{t^2} x = 0$ :

(i) Is oscillatory if $K > \frac{1}{4}$

(ii) Is non-oscillatory if $K \leq \frac{1}{4}$

9. (a) Let the matrix A in the system $x' = Ax$ where A is an $n \times n$ constant matrix and $x \in R^n$ be stable. Prove that for any solution of x(t) of the system $x' = Ax$

$$\lim_{t \to \infty} \|x(t)\| = 0$$

(b) Let the matrix A in $x' = Ax$ where A is an $n \times n$ constant matrix and $x \in R^n$ be stable. Prove that all solutions of $y' = Ay + f(t, y)$, $t \in J = [0, \infty]$ where $f \in C[J \times R^n, R^n]$ tend to

$x_n(t) = \int_a^b G\ (t, s)\ f(s, x_{n-1}(s))\ ds$, n = 1, 2, 3 ····· converges to a function x which is the unique solution of $x(t) = \int_a^b G\ (t, s)\ f(s, x(s))\ ds$. In addition, an upperbound on the error (due to truncation at the $n^{th}$ stage) is given by :

$$\|x_n - x\| < \frac{p^n}{1-p} \|x_1 - x_0\|$$

8. (a) Let $p(t) > 0$, $r_1(t)$, $r_2(t)$ and $p(t)$ be continuous functions on (a, b). Assume that x(t) and y(t) are real solutions of

$$(px')' + r_1 x = 0$$

$$(py')' + r_2 y = 0$$

respectively on (a, b). Further, let $r_2(t) \geq r_1(t)$ for $t \in (a, b)$. Prove that between any two consecutive zeros $t_1$, $t_2$ of x in (a, b) there exists at least one zero of y unless $r_1 = r_2$



4. (a) Find a fundamental matrix for $x' = Ax$ where

$$A = \begin{bmatrix} 3 & -2 \\ -2 & 3 \end{bmatrix}.$$

(b) Let f(t) be periodic with period w. Prove that a solution x(t) of $x' = Ax + f(t)$ is periodic of period w if and only if x(o) = x(w).

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5. (a) Let f, g, h be non-negative continuous functions defined for $t \in I$. Then prove that the inequality $f(t) \leq h(t) + \int_{t_0}^{t} g(s) f(s)\, ds$ $t \geq t_0$, $t \in I$ implies the inequality :

$$f(t) \leq h(t) + \int_{t_0}^{t} g(s) f(s)\, ds$$

(b) Let f(t, x) be continuous and be bounded for L and satisfy Lipschitz condition with Lipschitz constant K on the closed rectangle R. Prove that the successive approximations $x_n$, given by $x_n(t) = x_0 + \int_{t_0}^{t} f\left(s, x_{n-1}(s)\right) ds$,

$n = 1, 2, \ldots\ldots$ converge uniformly on the interval $I = ||t - t_0|| \le h = \min\left(a, \frac{b}{L}\right)$ to a solution x of the IvP $x' = f(t, x)$, $x(t_0) = x_0$. In addition this solution is unique.

6. (a) Let $v, w \in C'((t_0, t_0 + h), R)$ be lower and upper solutions of $x' = f(t, x)$, $x(t_0) = x_0$ respectively.

Suppose that, for $x \ge y$, f satisfies the inequality $f(t, x) - f(t, y) \le L(x - y)$ where L is a positive constant. Prove that $v(t_0) \le w(t_0)$ implies that $v(t) \le w(t)$, $t \in (t_0, t_0 th)$.

(b) Let $f, F \in C[I \times R^n, R^n]$ and let $\frac{\partial f}{\partial x}$ exist and be continuous on $I \times R^n$. Prove that if $x(t, t_0, x_0)$ is the solution of $x' = f(t, x)$, $x(t_0) = x_0$ existing for $t \ge t_0$ then any solution $y(t, t_0, x_0)$ of $y' = f(t, y) + F(t, y)$, $y(t_0) = x_0$ satisfies the integral equation

$$y(t, t_0, x_0) = x(t, t_0, x_0) + \int_{t_0}^{t} \overline{\Phi}(t, s, y'(s), t_0, x_0)$$

$F(s, y(s, t_0, x_0))\, ds$ for $t \ge t_0$ where $\Phi(t, t_0, x_0) = \frac{\partial x(t, t_0, x_0)}{\partial x_0}$.

7. (a) Assume that : (i) A, B are finite numbers and (ii) The functions $p'(t)$, $q(t)$ and $r(t)$ are real valued continuous functions on [A, B]. For the parameters $\lambda, \mu (\lambda \ne \mu)$ let x and y be the corresponding solutions of $(px')' + qx + \lambda rx = 0$. $A \le t \le B$ such that $[pw(x, y)]_A^B = 0$ where $w(x, y)$ is the Wronskion of x and y. Prove that :

$$\int_A^B r(s)x(s)\, y(s)\, ds = 0$$

(b) Assume that the function $f(t, x)$ in $x'' + f(b, x) = 0$, $x(a)\, x(b) = 0$, $a \le t \le b$ satisfies the Lipschitz condition $|f(t, x) - f(t, y)| < K|x - y|$ uniformly in t where K is a Lipschitz constant such that $p = \frac{K(b - a)^2}{8} < 1$.

Prove that the sequence $\{x_n\}$ defined by

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