10. (a) Let H be a Hilbert space and $A \in BL(H)$. Let A be self-adjoint. Then show that $\|A\| = \sup\{|\langle A(x), x\rangle| : x \in H, \|x\| \leq 1\}$. In particular, $A = 0$ iff $\langle A(x), x\rangle = 0$ for all $x \in H$.

(b) Let $K = \mathbb{C}$ and $A \in BL(H)$. Then prove that there are unique self-adjoint operators B and C on H such that $A = B + iC$. Further, A is normal iff $BC = CB$.

2013

Time : 4 hours

Full Marks : 100

The questions are of equal value.

Answer any five questions.

Symbols used have their usual meaning.

(FUNCTIONAL ANALYSIS)

1. (a) Let Y be a closed subspace of a normed space X. Let $(x_n + y)$ be a sequence in the quotient space X/Y with usual quotient norm. Then prove that the squence $(x_n + y)$ converges to $x + y$ in X/Y if and only if there is a sequence (y_n) in Y such that $(x_n + y_n)$ converges to x in X.

(b) Let X and Y be normed spaces and $F : X \to Y$ be a linear map such that the range

R(F) of F is finite dimentional. Then show that F is continuous if the zero space Z(F) is closed in X. Further, show that a linear functional F on X is continuous and only iff Z(F) is closed on X.

2. (a) Let X be a normed space over K, $f \in X'$ and $f \neq 0$. Let $a \in X$ with $f(a) = 1$ and $r > 0$. Then show that $U(a, r) \cap Z(f) = \phi$ if and only if $\|f\| \leq \frac{1}{r}$.

(b) Prove that a normed space X is a Banach space if and only if every absolutely summable series of elements in X is summable in X.

3. (a) Let X be a Banach space and Y be a normed space. Let (F_n) be a sequence in BL(X, Y). Let E be a totally bounded subset of X. Then show that $(F_n(x))$ converges to F(x) uniformly for $x \in E$ where $F \in BL(X, Y)$.

(b) State and prove Banach's open mapping theorem.

4. Show that the coefficient functional corresponding to a Schauder basis for a Banach space X are continuous. Infact they form a bounded subset of its dual X'.

5. (a) Let X and Y be normed spaces. Then show that for $F \in BL(X, Y)$:

$$\|F\| = \sup\{|y'(F(x))| : x \in X, \|x\| \leq 1, y' \in Y', \|y'\| \leq 1\}$$

(b) Let X, Y and Z be normed spaces. Then show that :

(i) $(F_1 + F_2)' = F_1' + F_2'$ and $(kF_1)' = kF_1'$ where $F_1, F_2 \in BL(X, Y)$ and $k \in K$.

(ii) $(GF)' = F'G'$, where $F \in BL(X, Y)$ and $G \in BL(Y, Z)$.

6. (a) Let X be a reflexive normed space. Then prove that :

(i) X is Banach and it remains reflexive in any equivalent norm.

(ii) X' is reflexive.

(b) Let (x'_n) be a sequence in a normed space X. If (x'_n) is bounded and $(x'_n(x))$ is a Cauchy sequence in K for each x in a subset of X, whose span is dense in X, then show that (x'_n) is weak convergent in X'. The converse holds if X is a Banach space.

7. (a) Let $H = K^n$. For $x = (x(1), \ldots, x(n))$ $y = (y(1), \ldots, y(n))$ in H, define

$$<x, y> = \sum_{j=1}^{n} x(j)\, \overline{y(j)}.$$ Then show that

$<, >$ is an inner product on H.

(b) Let $\{u_\alpha\}$ be an orthonormal set in an inner product space X and $x \in X$. Let $E_x = \{u_\alpha : <x, u_\alpha> \neq 0\}$. Then show that E_x is a countable set. If E_x is denumerable, then $< x, u_n> \to 0$ as $n \to \infty$.

8. (a) Let X be an inner product space. Let $E \subset X$ and $X \in \bar{E}$. Then show that there exists a best approximation form E to x iff $x \in E$. Further, if $E \subset X$ is convex . Then there exists at most one best approximation from E to any $x \in X$.

(b) Let H be a Hilbert space and F be a non-empty closed subspace of H. Then $H = F + F^1$. Moreover $F^{11} = F$.

9. (a) Let H be a Hilbert space and $A \in BL(H)$. Let $Z(A) = R(A^*)^1$ and $Z(A^*) = R(A)^1$. Then show that A is injective if and only if $R(A^*)$ is dense in H and A^* is injective iff R(A) is dense in H. Also closure of $R(A) = Z(A^*)^1$ and closure of $R(A^*) = Z(A)^1$.

(b) Let $A \in BL(H)$ and F be a closed subspace of H. Then show that $A(F) \subset F$ iff $A^*(F^1) \subset F^1$ and in that case $(A_{|F})^* = PA^*_{|F}$, where P is the orthogonal projection of H onto F.