9. (a) Prove that an elliptic function without poles is a constant.

(b) Prove that :

$$\begin{vmatrix} P(z) & P'(z) & 1 \\ P(u) & P'(u) & 1 \\ P(u+z) & -P'(u+z) & 1 \end{vmatrix} = 0$$

10. (a) Show that any elliptic function with periods w_1, w_2 can be written as

$$C \prod_{k=1}^{n} \frac{\sigma(z-a_k)}{\sigma(z-b_k)}, \text{ where C is a constant.}$$

(b) Prove that a non-constant elliptic function has equally many poles as it as zeros.

2014

Time : 4 hours

Full Marks : 100

The questions are of equal value.

*Answer any **five** questions.*

The symbols used have their usual meanings.

(COMPLEX ANALYSIS)

1. (a) Show that z and z′ correspond to diametrically opposite points on the Riemann sphere if and only if $z z^{-1} = -1$.

(b) If $|a_i| < 1$, $\lambda_i \geq 0$ for $i = 1, 2, \dots\dots\dots, n$ and $\lambda_1 + \lambda_2 + \dots\dots\dots + \lambda_n = 1$, then show that $|\lambda_1 a_1 + \lambda_2 a_2 + \dots\dots\dots + \lambda_n a_n| < 1$.

2. (a) Show that a harmonic function satisfies the formal differential equation $\frac{\partial u}{\partial z \partial \bar{z}} < 0$.

(b) Expand $\frac{2z+3}{z+1}$ in powers of z – 1. What is the radius of convergence ?

3. (a) Prove that a linear transformation carries circles into circles.

(b) Suppose that f(z) is analytic and satisfies the condition $|f(z)^2 - 1| < 1$ in a region Ω. Show that either Re f(z) > 0 or Re f(z) < 0 throughout Ω.

4. (a) If a function f(z) is analytic on R, then show that $\int_{\partial R} f(z)\,dz = 0$.

(b) If a piecewise differentiable closed curve C does not pass through the point a, then the value of the integral $\int_C \frac{dz}{z-a}$ is a multiple of $2\pi i$.

5. (a) Find the poles and residues of :

(i) $\frac{1}{(z^2-1)^2}$

(ii) cot z

(b) If f(z) is analytic in Ω, then $\int_C f(z)\,dz = 0$ for every cycle C which is homologous to zero in Ω.

6. (a) Compute $\int_0^{2\pi} \frac{\cos 2\theta}{5+4\cos\theta}\,d\theta$

(b) Evaluate $\int_0^{\infty} \frac{dx}{x^4+a^4}$ by the method of residues.

7. (a) Develop log (sin z/z) in powers of z upto the terms z^6.

(b) Show that :

$$\Gamma\left(\frac{1}{6}\right) = 2^{\frac{-1}{2}}\left(\frac{3}{\pi}\right)^{\frac{1}{2}}\Gamma\left(\frac{1}{3}\right)^2$$

8. (a) Show that an entire function of fractional order assumes every finite value infinetely many times.

(b) Show that :

$$\xi(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\,\xi(1-s)$$