

2014

Time : 4 hours

Full Marks : 100

The questions are of equal value.

Answer any **five** questions from each Group.

Symbols used have their usual meanings.

**(ADVANCED CALCULUS AND PARTIAL DIFFERENTIAL EQUATION)**

Group – A

Marks : 50

**(ADVANCED CALCULUS)**

1. Let  $f \in C'$  in an open ball  $B(p_0, r)$  about the point  $p_0$  in  $n$ -space. Let  $p \in B$  and set

$$p - p_0 = \Delta p = (\Delta x_1, \Delta x_2, \dots, \Delta x_n).$$

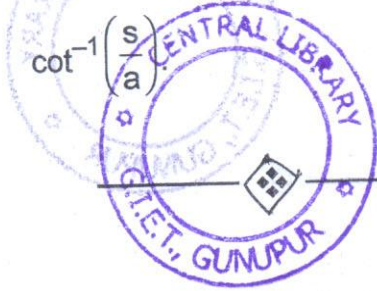
Then, show that there are points

$p_1, p_2, \dots, p_n$  in  $B$  such that

(a) Find the Laplace transform of the function  $f(t) = e^{at} \sqrt{t}, a > 0.$

(b) Find the inverse Laplace transform of

$$\cot^{-1}\left(\frac{s}{a}\right)$$



$$\int_{\partial D} \omega = \iint_D d\omega.$$

$$+ \begin{matrix} 1, 0, -1 \\ \sqrt{2} \\ 2/2 \end{matrix}$$

Very =

$$f(p) - f(p_0) = f_1(p_1) \Delta x_1 + f_2(p_2) \Delta x_2 + \dots + f_n(p_n) \Delta x_n$$

2. Transform the differential equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

into polar coordinates by the substitution  $x = r \cos \theta, y = r \sin \theta$ .

3. Show that for any  $p \in S$  and any  $u \in \mathbb{R}^3$ ,  $\frac{dg}{du}(u) =$

$$Dg(p) \cdot u.$$

4. Let  $T$  be a transformation of class  $C'$  on an open set  $D$  in  $n$ -space and let  $E$  be a compact set in  $D$ . Then there are numbers  $M$  and  $\delta > 0$  such that  $|T(p) - T(q)| \leq M |p - q|$  for all  $p$  and  $q$  in  $E$  with  $|p - q| < \delta$ . If  $E$  is convex, then  $\delta = \text{diam}(E)$ .

5. Let  $T$  be a transformation from  $\mathbb{R}^3$  into  $\mathbb{R}^3$  described by

$$u = f(x, y, z)$$

$$v = g(x, y, z)$$

$$w = h(x, y, z)$$

which is of class  $C'$  in an open set  $D$  and suppose that at each point  $p \in D$  the differential  $dT$  has rank 2. Then show that  $T$  maps  $D$  onto a surface in  $UVW$  space and the functions  $f, g$  and  $h$  are functionally dependent in  $D$ .

6. Let  $D$  be the region in the first quadrant which is bounded by the curves  $xy = 1, xy = 3, x^2 - y^2 = 1, x^2 - y^2 = 4$ . Make an appropriate substitution and evaluate  $\iint_D (x^2 + y^2) dx dy$ .

7. Define smoothly equivalent and show that if  $\gamma_1$  and  $\gamma_2$  are smoothly equivalent curves, then  $L(\gamma_1) = L(\gamma_2)$ .

8. Find the area of the portion of the upper hemisphere of the sphere with centre  $(0, 0, 0)$  and radius  $R$  that obeys  $x^2 + y^2 - Rz \leq 0$ .

9. If  $w$  is a differential form of class  $C'$ , then show that  $T^*(dw) = d(w)^* = d(w^*) = d T^*(w)$ .

10. State and prove Green's theorem.

Group - B

Marks : 50

(PARTIAL DIFFERENTIAL EQUATION)

11. Find the general solution of  $u_{xy} + u_x = 0$  by setting  $u_x = v$ .

12. Obtain the general solution of

$$3u_{xx} + 10u_{xy} + 3u_{yy} = 0.$$

13. Determine the solution of

$$u_{xx} - u_{yy} = 1;$$

$$u(x, 0) = \sin x;$$

$$u_y(x, 0) = x.$$

14. Determine the eigenvalues and eigenfunctions of Sturm-Liouville system

$$u'' + \lambda u = 0; 0 \leq x \leq \pi$$

$$u(0) = 0; u'(\pi) = 0.$$

15. Show that the eigenfunctions of a periodic Sturm-Liouville system in  $[a, b]$  are orthogonal with respect to the weight functions in  $[a, b]$ .

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(4)

Contd.

16. Show that the Green's function for the boundary value problem is symmetric.

17. (a) Find the Fourier sine transform of  $e^{-ax}$  for  $a > 0$ .

(b) If  $F(f)$  is the Fourier transform of a function  $f$ , then show that  $F(f(t-c))$  is  $e^{-iac} F(f)$ , where  $c$  is a real constant.

18. Using Fourier transform obtain the solution of the initial value problem of heat conduction in an infinite rod :

$$u_t - u_{xx} = 0, -\infty < x < \infty, t > 0$$

$$u(x, 0) = f(x), -\infty < x < \infty$$

$u(x, t)$  is bounded

19. Define convolution in Laplace transform and show that :

(i)  $f * g = g * f$

(ii)  $f * (g * h) = (f * g) * h$

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(5)

(Turn over)