20. (a) Find the Laplace transform of the function f(t) $= e^{at}\sqrt{t}$, $a > 0$.

(b) Find the inverse Laplace transform of $\cot^{-1}\left(\frac{s}{a}\right)$.

2014

Time : 4 hours

Full Marks : 100

The questions are of equal value.

*Answer any **five** questions from each Group.*

Symbols used have their usual meanings.

(ADVANCED CALCULUS AND PARTIAL DIFFERENTIAL EQUATION)

Group – A

Marks : 50

(ADVANCED CALCULUS)

1. Let $f \in C'$ in an open ball $B(p_o, r)$ about the point p_o in n-space. Let $p \in B$ and set

$$p - p_o = \Delta p = (\Delta x_1, \Delta x_2, \ldots, \Delta x_n).$$

Then, show that there are points

$p_1, p_2, \ldots, p_n$ in B such that

$f(p) - f(p_o) = f_1(p_1)\,\Delta x_1 + f_2(p_2)\,\Delta x_2 + \ldots\ldots + f_n(p_n).\,\Delta x_n$

2. Transform the differential equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$ into polar coordinates by the substitution $x = r\cos\theta,\ y = r\sin\theta$.

3. Show that for any $p \in S$ and any $u \in R^3$, $dg_p(u) = Dg(p).\,u$.

4. Let T be a transformation of class C′ on an open set D in n-space and let E be a compact set in D. Then there are numbers M and $\delta > 0$ such that $|T(p) - T(q)| \le M\,|p - q|$ for all p and q in E with $|p - q| < \delta$. If E is convex, then $\delta = \text{dia m}(E)$.

5. Let T be a transformation from R^3 into R^3 described by

$$u = f(x, y, z)$$

$$v = g(x, y, z)$$

$$w = h(x, y, z)$$

which is of class C′ in an open set D and suppose that at each point $p \in D$ the differential dT has rank 2. Then show that T maps D onto a surface in UVW space and the functions f, g and h are functionally dependent in D.

6. Let D be the region in the first quadrant which is bounded by the curves $xy = 1$, $xy = 3$, $x^2 - y^2 = 1$, $x^2 - y^2 = 4$. Make an appropriate substitution and evaluate $\iint_D (x^2 + y^2)\,dx\,dy$.

7. Define smoothly equivalent and show that if γ_1 and γ_2 are smoothly equivalent curves, then $L(\gamma_1) = L(\gamma_2)$.

8. Find the area of the portion of the upper hemisphere of the sphere with centre (0, 0, 0) and radius R that obeys $x^2 + y^2 - Ry \le 0$.

9. If w is a differential form of class C′, then show that $T^*(dw) = d(w)^* = d(w^*) = d\,T^*(w)$.

10. State and prove Green's theorem.

Group – B

Marks : 50

(PARTIAL DIFFERENTIAL EQUATION)

11. Find the general solution of $u_{xy} + u_x = 0$ by setting $u_x = v$.

12. Obtain the general solution of

$$3u_{xx} + 10u_{xy} + 3u_{yy} = 0.$$

13. Determine the solution of

$$u_{xx} - u_{yy} = 1 \,;$$
$$u(x, 0) = \sin x \,;$$
$$u_y(x, 0) = x.$$

14. Determine the eigenvalues and eigenfunctions of Sturm-Liouville system

$$u'' + \lambda u = 0 \,;\, 0 \le x \le \pi$$
$$u(0) = 0 \,;\, u'(\pi) = 0.$$

15. Show that the eigenfunctions of a periodic Sturm – Liouville system in [a, b] are orthogonal with respect to the weight functions in [a, b].

16. Show that the Green's function for the boundary value problem is symmetric.

17. (a) Find the Fourier sine transforms of $f(x) = e^{-ax}$ for $a > 0$.

(b) If $F(f)$ is the Fourier transform of a function f, then show that $F(f(t - c))$ is $e^{iac} F(f)$, where c is a real constant.

18. Using Fourier transform obtain the solution of the initial value problem of heat conduction in an infinite rod :

$$u_t - u_{xx} = 0, \; -\infty < x < \infty, \; t > 0$$
$$u(x, 0) = f(x), \; -\infty < x < \infty$$
$$u(x, t) \text{ is bounded}$$

19. Define convolution in Laplace transfrom and show that :

(i) $f * g = g * f$

(ii) $f * (g * h) = (f * g) * h$