$\|\phi_n\| \leq \alpha, \forall n \geq 1$, where Y is a Banach space. If $\langle \phi_n(x) \rangle$ converges for every $x \in E$, then for some $\phi \in BL(X, Y)$, $\phi_n(x) \to \phi(x)$ for all $x \in X$.

10. (a) If T is a normal operator on Hilbert space H, then $\|T^2\| = \|T\|^2$.

(b) Let U be a linear operator on a Hilbert space V. Then the following are equivalent.

(i) $U^* = U^{-1}$.

(ii) $UU^* = U^*U = I$

(iii) $\langle U(u), U(v) \rangle = \langle u, v \rangle \ \forall\, u, v \in V$

(iv) $\|U(u)\| = \|u\| \ \forall\, u \in V$

2015

Time : 4 hours

Full Marks : 100

The questions are of equal value.

Answer any ***five*** *questions.*

Symbols used have their usual meanings.

(FUNCTIONAL ANALYSIS)

1. (a) Prove that every linear map from a finite dimensional normed linear space to any normed linear space in continuous.

(b) Let $\phi : \mathbb{R}^2 \to \mathbb{R}^2$. For $x = (x_1, x_2) \in \mathbb{R}^2$, let $\|x\| = \left(|x_1|^2 + |x_2|^2\right)^{\frac{1}{2}}$ and define

$\phi(x) = (x_1\cos\theta - x_2\sin\theta, x_1\sin\theta + x_2\cos\theta)$

Check whether :

(i) ϕ in Linear ?

(ii) ϕ in bounded ?

Find $\|\phi\|$.

2. (a) Let X in a normed linear space over $k(\mathbb{R} | \mathbb{C})$, Y in a subspace of X. If $g \in y'$, then $\exists f \in X'$ such that $f/y = g$ and $||g|| = ||f||$.

(b) Let $X_1, X_2 \ldots\ldots X_n$ be normed space, and $X = X_1 \times X_2 \times \cdots \times X_n$. Then X is a Banach space iff each X_k, $k = 1, 2 \cdots n$ is a Banach space.

3. (a) If E is a totally bounded subset of X, then prove that the sequence $<\phi_n(x)>$ converges to $\phi(x)$ unifomly.

(b) Let X, Y are Banach space. Suppose $\phi: X \rightarrow Y$ in a linear continuous map and ϕ is also onto. Then prove that ϕ is an open map.

4. (a) Let $X = \ell^p$ $(1 < p \leq \infty)$ with $||x|| = \left\{\sum |x_p|^p\right\}^{\frac{1}{p}}$.

Define $A : X \rightarrow X$ by $A(x_1, x_2, \ldots) = \left(x_1, \frac{x_2}{2}, \frac{x_3}{3}, \ldots\right)$. Then show that $A \in BL(X)$

(b) Prove that the eigen vectors corresponding to distinct eigen values are linearly dependent.

5. Show that the dual of $\mathbb{R}^n$ is isometrically isomorphic to $\mathbb{R}^n$.

6. (a) Show that a finite dimensional normed linear space convergence (strong) and weak convergence are equivalent.

(b) Prove that every finite dimensional normed linear space is reflexive.

7. Let X : Hilbert space and $\{u_n : n \geq 1\}$ be a countable orthonormal set in X.

For a sequence of scalars $<k_n>$ belongs to k, the following results are equivalent :

(a) $\exists x \in X$ s.t. $<x, u_n> = k_n \ \forall n$

(b) $\Sigma |k_n|^2 < \infty$

(c) $\sum_{n=1}^{\infty} k_n u_n$ converges in X.

8. State and prove Riesz representation theorem.

9. (a) A linear operator on $\mathbb{R}^2$ is defined by $T(x, y) = (x + 2y, x - y)$. Find the adjoint. If $\alpha = (1, 3)$, find $T^*(\alpha)$.

(b) Let E is a subset of X such that span (E) is dense in X. Suppose $\phi_n \in BL(X, Y)$ such that