Show that $\max_{|z|=r} |f(z)| \leq$ the maximum modulus of g and minimum modulus of f is $\geq$ the minimum modulus of g.

9. (a) Prove that the zeros $a_1, \cdots\cdots, a_n$ and poles $b_1, \cdots\cdots, b_n$ of an elliptic function satisfy $a_1 + \cdots\cdots + a_n = b_1 + \cdots\cdots + b_n \pmod{M}$.

   (b) What is unimodular transformation ? Show that any two bases of the same module are connected by a unimodular transformation.

10. (a) Show that any even elliptic function with periods w_1, w_2 can be expressed in the form $C \prod_{n=1}^{n} \frac{\gamma(z) - \gamma(a_k)}{\gamma(z) - \gamma(b_k)}$ (C = const) provided that 0 is neither a zero nor a pole. The symbol γ stands for Weierstrass Pe function.

    (b) Explain what do you mean by a modular function. How conformal mapping is effected by this function ?

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2015

Time : 4 hours

Full Marks : 100

The questions are of equal value.

*Answer any **five** questions.*

Symbols used have their usual meanings.

(COMPLEX ANALYSIS)

1. (a) Solve the quadratic equation :

   $z^2 + (\alpha + i\beta)z + \gamma + i\delta = 0.$

   (b) Simplify $\sin\phi + \sin 2\phi + \cdots\cdots + \sin n\phi$ and $1 + \cos\phi + \cos 2\phi + \cdots\cdots + \cos n\phi$.

2. (a) Develop $\frac{1}{z(z+1)^2 (z+2)^3}$ in partial fractions.

   (b) State and prove Cauchy's necessary and sufficient conditon of uniform convergence.

3. (a) Give a precise definition of a single valued branch of $\sqrt{1+z}+\sqrt{1-z}$ in a suitable region and prove that it is analytic.

(b) If z_1, z_2, z_3, z_4 are distinct points in the extended plane and T any linear transformation, then $\left(T_{z_1}, T_{z_2}, T_{z_3}, T_{z_4}\right) = (z_1, z_2, z_3, z_4)$.

4. (a) The line integral $\int_r pdx + qdy$, defined in Ω, depends only on the end points of γ if and only if there exists a function $U(x, y)$ in Ω with the partial derivatives $\frac{\delta u}{\delta x} = p$, $\frac{\delta u}{\delta y} = q$. Prove this.

(b) If $f(z)$ is analytic and $\operatorname{Im} f(z) \geq 0$ for $\operatorname{Im} z > 0$, show that $\frac{|f(z) - f(z_0)|}{|f(z) - f(z_0)|} \leq \frac{|z - z_0|}{|z - z_0|}$.

5. (a) Define locally exact differentials. If $pdx + qdy$ is locally exact in Ω, then prove that $\int_r pdx + qdy = 0$ for every cycle $\gamma \sim 0$ in Ω.

(b) Define simple connected region. Prove that the region obtained from a simply connected region by removing m points has the connectivity m + 1 and find a homology basis.

6. (a) Find out poles and residues of the functions $\frac{1}{z^2 + 5z + 6}$ and cot z.

(b) State and prove Rouche's theorem.

7. (a) The Legendre polynomials are defined as the coefficients $P_n(\alpha)$ in the development $(1 - 2\alpha z + z^2)^{-\frac{1}{2}} = 1 + P_1(\alpha)\, z + P_2(\alpha)\, z^2 + \ldots\ldots$ Find P_1, P_2, P_3 and P_4.

(b) Express $\sum_{-\infty}^{\infty} \frac{1}{z^3 - n^3}$ in closed form.

8. (a) Show that the function $\zeta(s) = \frac{1}{2}s\,(1 - s)\pi^{-s/2}\,\Gamma(s/2)\,\zeta(s)$ is entire and satisfies $\zeta(s) = \zeta(1 - s)$.

(b) Compare $f(z) = z^m \prod_n \left(1 - \frac{z}{a_n}\right)$ having genus zero with $g(z) = z^m \prod_n \left(1 - \frac{z}{|a_n|}\right)$.