9. (a) If $T \in A(v)$ has all its characteristic roots in F, then prove that there is a basis of V in which the matrix of T is triangular.

(b) Prove that the matrix $\begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 1 & 1 & 0 \end{pmatrix}$ is nilpotent, and find the invariants and Jordan form.

10. (a) Define trace of a matrix.

For $A, B \in F_n$ and $\lambda \in F$, show that :

(i) $tr(\lambda A) = \lambda \, tr\, A$

(ii) $tr(A + B) = trA + trB$

(iii) $tr(AB) = tr(BA)$

(b) For $A, B \in F_n$, prove that det (AB) = (det A) (det B).

2015

Time : 4 hours

Full Marks : 100

The questions are of equal value.

*Answer any **five** questions.*

Symbols used have their usual meanings.

(ALGEBRA)

1. (a) Define automorphism. If G is a group, then prove that A(G), the set of automorphisms of G, is also a group.

(b) Prove that there is no such a such that $a^{-1}(1, 2, 3)a = (1, 3)(5, 7, 8)$.

2. (a) If G is a finite group, then prove that $C_a = 0(G)/0(N(a))$.

(b) If $x > 0$ is a real number, define [x] to be m where m is an integer such that $m \leq x \angle m + 1$. If p is a prime, show that the power of p which exactly divides n ! is given by $\left[\frac{n}{p}\right]+\left[\frac{n}{p^2}\right]+\cdots\cdots+\left[\frac{n}{p^r}\right]+\cdots\cdots$.

3. (a) Let R be a commutative ring with unit element whose only ideals are (0) and R itself. Then prove that R is a field. Define Euclidean ring, its unit and associates of elements. If R be a Euclidean ring and $a, b \in R$ and if $b \neq 0$ is not a unit in R, then prove that $d(a) < d(ab)$.

(b) State and establish the division algorithm in polynomial rings.

4. (a) Define internal direct sum and external direct sum. If V is the internal direct sum of $u_1, \cdots\cdots u_n$, then show that V is isomorphic to the external direct sum of $u_1, \cdots\cdots u_n$.

(b) Prove that e is transendental.

5. (a) If p(x) is a polynomial in F[x] of degree $n \geq 1$ and is irreducible over F, then prove that there is an extension E of F, such that $[E : F] = n$, in which p(x) has a root.

(b) If F is of characteristic 0 and if a, b are algebraic over F, then prove that there exists an element $c \in F(a, b)$ such that $F(a, b) = F(c)$.

6. (a) Show that Hom (v, w) is a vector space over F under suitable operations.

(b) Define inner product space. If $u, v \in V$, where V is inner product space, then prove that $|(u, v)| \leq \|u\|\,\|v\|$.

7. (a) Define the terms 'normal extension of a field', 'splitting field'. Prove that K is a normal extension of F if and only if K is the splitting field of some polynomial over F.

(b) Prove that S_4 is a solvable group.

8. (a) If A is an algebra, with unit element, over F, then prove that A is isomorphic to a subalgebra of A(v) for some vector space V over F.

(b) If $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & -11 & 6 \end{pmatrix} \in F_3$, then prove that $A^3 - 6A^2 + 11A - 6 = 0$.