4. (*a*) A Banach space can't have a denumerable basis.

OR

(*b*) (*i*) Let X and Y be normed spaces and $X \neq \{0\}$. Then $BL(X, Y)$ is a Banach space in the operator norm iff Y is Banach space.

(*ii*) Prove that the dual X' of every normed space X is a Banach space.

5. (*a*) State and prove that bounded inverse theorem.

OR

(*b*) State the closed graph theorem.

6. (*a*) Let x be a normed space and $A \in BL(x)$ be a finite rank. Then prove that $\sigma_e(A) = \sigma_a(A) = \sigma(A)$.

OR

(*b*) Let $1 \leq p \leq \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then prove that the following :

(*i*) The dual of K^n with the norm $\| \|_p$ is linearly isometric to K^n with the norm $\| \|q$.

(*ii*) The dual of C_{00} with the norm $\| \|_p$ is linearly isometric to l^q.

(*iii*) The dual of C_0 with the norm $\| \|_\infty$ is linearly isometric to l'.

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2016

FUNCTIONAL ANALYSIS - I

Time : Three Hours] [*Maximum Marks* : 80

The figures in the right hand margin indicate marks. Answer from both the Sections as directed.

SECTION-A

1. Answer any **four** of the following : 4×4

(*a*) Let X be a normed space such that $\bar{U}(0,1)$ is totally bounded. Then show that X is finite dimensional.

(*b*) Let X and Y be normed spaces and $F : X \to Y$ be a linear map. Prove that the following conditions are equivalent :

(*i*) F is bounded on $\bar{U}(0,r)$ for some $r > 0$

(*ii*) F is continuous at 0

(*iii*) F is continuous on X

(*c*) Let X be a linear space over K and Y be a subspace of X which is not a hyperspace in X. If x_1 and x_2 are in X but not in Y, then there is some x in X such that for all $t \in [0,1]$,

$tx_1 + (1-t)x \notin Y$ and $tx_2 + (1-t)x \notin Y$

If X is a normal space, then Y^c is connected. Justify.

(*d*) A normed space X is a Banach space iff every absolutely summable series of elements in X is summable in X. Justify.

(*e*) State and prove the Resonance theorem.

(*f*) Let X be a Banach space, $A \in BL(X)$ and $\| A^p \| < 1$ for some positive integer p. Then prove that the bounded operator $I - A$ is invertible. Also,

$(I-A)^{-1} = \sum A^n$ and $\| (I-A)^{-1} \|$

$$\leq \frac{1+\|A\|+...+\|A^{p-1}\|}{1-\|A^p\|}$$

OR

2. Answer **all** questions from the following : 2×8

(*a*) Definite Jenson's inequality.

(*b*) Define operator norm.

(*c*) Let X be a linear space over c. A real value linear functional $u : X \to \mathbb{R}$. Define $f(x) = u(x) - iu(ix)$, $x \in X$. Then show that f is a complex-linear functional on X.

(*d*) Let X be a normed space over K, $j \in X'$ and $f \neq 0$. Let $a \in X$ with $f(a) = 1$ and $r > 0$. Then

$\cup(a,r) \cap z(f) = \phi$ iff $\|f\| \leq \frac{1}{r}$

Justify.

(*e*) Let X and Y be normed spaces and $F : X \to Y$ be linear. Then prove that F is continuous $\Leftrightarrow g \circ f$ is continuous for every $g \in Y^1$.

(*f*) If Z is a closed subspace of normed space X, then quotient map Q from X to $\frac{X}{Z}$ is continuous and open. Prove it.

(*g*) Prove that the set of all invertible operators is open in $BL(X)$ and the map $A \to A^{-1}$ is continuous on this set.

(*h*) State Resonance Theorem's converse part.

SECTION-B

Answer **all** questions : 16×4

3. Let X be a normed space. Then show that

(*a*) If E_1 is open in X and $E_2 \subset X$, then $E_1 + E_2$ is open in X.

(*b*) Let E be a convex subset of X. Then the interior E^0 of E and the closure $\overline{E}$ of E are also convex. If $E^0 \neq \phi$, then $\overline{E} = \overline{E}_0$.

(*c*) Let Y be a subspace of X. Then $Y^0 \neq \phi$ iff $Y = X$.

OR

(*a*) Let X be a normed space and f be a non-zero linear functional on X. Then f is discontinuous iff $z(f)$ is dense in X. Prove it.

(*b*) Let X and Y be normed spaces and $F : X \to Y$ be linear. Then F is continuous iff for every Cauchy sequence $\langle x_n \rangle$ in X, the sequence $\langle Fx_n \rangle$ is Cauchy in Y.