(b) A necessary and sufficient condition that a linear operator T on a complex Hilbert space V (Unitary Space) be self-adjoint (or Hermitian) is that $\langle T(\alpha), \alpha \rangle$ be real for every α.

2016

Time : 4 hours

Full Marks : 100

The questions are of equal value.

*Answer any **five** questions.*

Symbols used have their usual meanings.

(FUNCTIONAL ANALYSIS)

1. (a) Let X and Y are linearly homeomorphic. X is complete iff Y is complete.

   (b) Let $\phi : \ell^\infty \to \ell^\infty$ defined by $x = (x_n) \to \left(\frac{x_n}{n}\right)$.

   Then ϕ is linear, bounded. Find $\|\phi\|$.

2. (a) Prove that a Banach space cann't have a denumerable basis.

   (b)Let $X = \mathbb{R}^2$ with the suprimum norm. Consider the subspace $Y = \langle (x_1, x_2) \in X : x_1 = x_2 \rangle$, $g : Y \to R$ by $g(x_1, x_2) = x_1$. Define $f_1 : X \to R$ by

$f_1(x_1, x_2) = x_1$ and $f_2 : X \to R$ by $f_2(x_1, x_2) = x_2$. Show that f_1 and f_2 are Hahn-Banach extension to X.

3. (a) Let X and Y are Banach Space. Then prove that $X \times Y$ a Banach space.

(b) Let X and Y are Banach Spaces. $\phi : X \to Y$ is a linear map then $G(\phi)$ is closed iff ϕ is continuous.

4. (a) Let $A \in BL(X)$ is invertible and $\| B - A \| < \frac{1}{\|A^{-1}\|}$, then prove that B is invertible, where BL(X) is the set of the all bounded linear functional on X to X.

(b) Prove that the eigen vectors corresponding to distinct eigen values are linearly dependent.

5. Show that $(l_p)' \simeq (l_q)$ where $\frac{1}{p} + \frac{1}{q} = 1$ $(1 \le p < \infty)$.

6. (a) If X is reflexive, then weak convergence iff weak * convergence.

(b) Prove that every finite dimensional normed linear space is reflexive.

7. (a) State and prove Gram-Schmidt orthonormalization process.

(b) Prove that a normed linear space is an inner product space if parallelogram law is true.

8. (a) Let F is a non-empty closed subspace of a Hilbert space X. For $x \in X$, $\exists$ a unique $y \in F$ such that $\| x - y \| = \inf_{z \in f} \| x - z \|$.

(b) Let X be a Hilbert space and F be a non-empty closed subspace of X. Then prove that $X = F \oplus F^1$.

9. (a) A linear operator on $\mathbb{R}^2$ is defined by $T(x, y) = (x + 2y, x - y)$. Find the adjoint. If $\alpha = (1, 3)$, find $T * (\alpha)$.

(b) Let E is a subset of X such that span(E) is dense in X. Suppose $\phi_n \in BL(X, Y)$ such that $\| \phi_n \| \le \alpha$, $\forall\ n \ge 1$, where Y in a Banach space. If $<\phi_n(x)>$ converges for every $x \in E$, then for some $\phi \in BL(x, y)$, $\phi_n(x) \to \phi(x)$ for all $x \in X$.

10. (a) Let T be a linear operator on a complex Hilbert space V. Then T is normal $\Leftrightarrow \| T^*(u) \| = \| T(u) \| \ \forall\ u \in V$.