2016
(January)

Time : 3 hours

Full Marks : 80

The figures in the right-hand margin indicate marks.

Answer from both the Sections as directed.

The symbols used have their usual meanings.

(ELEMENTARY COMPLEX ANALYSIS)

Section – A

1. Answer any **four** of the following : $4 \times 4 = 16$

(a) Express $\cos 3\phi$ in terms of $\cos\phi$ and $\sin\phi$.

(b) Give ϵ δ definition of a continuous function. Using this prove that $f(z) = \bar{z}$ and $f(z) = |z|$ are continuous functions.

(c) Show that if $f(z) = u(x, y) + i\,u(x, y)$ is analytic then $|f'(z)|^2$ is the Jaccobian of u, v with respect to x, y.

(d) Define the terms : Parallel translation and Homothetic transformation. Find the image of $|z| = 1$ under these transformations.

(e) State and prove Morera's theorem.

(f) Compute $\int_{|z|=1} \frac{e^z}{z^n}\, dz$.

OR

2. Answer all questions from the following : 2×8 = 16

(a) Express – 6 in polar form.

(b) Find the real part of $\frac{1+z}{1-z}$.

(c) Verify Cauchy-Riemann equations for $f(z) = z^2$.

(d) Find the radius of convergence of the series. $\sum (3+(-1^n)^n z^n$.

(e) Give example of a function which is note a polynomial and is analytic in the whole plane.

(f) Compute $\int_\gamma x\, dz$ where γ is the directed line segment from 0 to 1 + i.

(g) Define an exact differential in a region.

(h) Find the value of the cross ratio (o, i, I, ∞).

Section – B

Answer all questions of the following : 16×4 = 64

3. (a) (i) State and prove triangle inequality for complex numbers. Derive necessary and sufficient conditions for equality.

(ii) Find symmetric point of a with respect to the lines which bisect the angles between the coordinate axes.

OR

(b) (i) State and proof Cauchy's inequality for complex numbers. Derive necessary and sufficient conditions for equality.

(ii) If $|a_k| < 1$, $\lambda_k \geq 0$ for k = 1, z, ⋯, n and $\sum_{k=1}^{n} \lambda_k = 1$, Show that $\left|\sum_{k=1}^{n} \lambda_k a_k\right| < 1$.

4. (a) (i) If $g(w)$ and $f(z)$ are analytic, prove that $g(f(z))$ is also analytic.

(ii) State and prove Lucas theorem.

OR

(b) Prove the following :

For every power series $\sum a_x z^n$ there exists a number R, $0 \le R \le \infty$, with the following properties.

(i) The series converges absolutely for $|z| < R$. If $0 < P < R$, the convergence is uniform for $|z| \le f$.

(ii) If $|z| < R$ the terms of the series are unbounded and series is consequently divergent.

(iii) In $|z| < R$, the sum of the series is an analytic function. The derivative can be obtained by term by term differentiation, and the derived series has same radius of convergence.

5. (a) (i) Prove that if z_1, z_2, z_3, z_4 are distinct points in the extended plane and T is any Möbius transformation the $(Tz_1, Tz_2, Tz_3, Tz_4) = (z_1, z_2, z_3, z_4)$.

(ii) Prove that the cross ratio (z_1, z_2, z_3, z_4) is real if and only if the four points lie on a circle or straight line.

OR

(b) Prove the following :

An analytic function in a region Ω whose derivative vanishes identically must reduce to a constant. The same is true if either the real part, the imaginary part, the modulus or the argument is constant.

6. (a) State and prove Cauchy's integral formula for circular disc. Derive integral formula for higher derivatives of an analytic function in a region. Compute $\int_{|z|=2} \frac{(1-z)^4}{z^6} dz$.

OR

(b) State and prove the Cauchy's general theorem in its homological form.