equality, addition and scalar multiplication are defined component wise. Prove that V is a vector space over $\mathbb{R}$.

6. (a) (i) Show that the two extension fields $Q\left(\sqrt{2}+\sqrt{3}\right)$ and $Q\left(\sqrt{2}, \sqrt{3}\right)$ over Q are same.

(ii) Prove that a polynomial of degree n over a field can have at most n roots in any extension field.

OR

(b) (i) Find the splitting field of the polynomial $x^5 - 2$ over Q.

(ii) If the field F is of characteristic 0 and if a, b are algebraic over F, then prove that there exists an element $c \in F(a, b)$ such that $F(a, b) = F(c)$.

2016

(January)

Time : 3 hours

Full Marks : 80

The figures in the right-hand margin indicate marks.

Answer from both the Sections as directed.

The symbols used have their usual meanings.

(ALGEBRA – I)

Section – A

1. Answer any **four** of the following : $4 \times 4 = 16$

(a) For g in a group G, define $T_g : G \to G$ by $xT_g = g^{-1}xg$, for $x \in G$. Prove that T_g is an automorphism.

(b) Prove that $N(a)$, the normalizer of $a \in G$ is a subgroup of G.

(c) Show that a group G of order 30 has a normal subgroup of order 15.

(d) If the vector spaces V and W are isomorphic under the isomorphism T, then show that T maps a basis of V onto a basis of W.

(e) Prove that the minimal polynomial satisfied by an algebraic element over F is irreducible.

(f) If 'a' is an element of the Euclidean ring R show that d(a) = d(1) if and only if 'a' is a unit in R.

OR

2. Answer **all** questions from the following : $2 \times 8 = 16$

(a) If φ is an automorphism on a finite group G and $a \in G$, then show that order of 'a' and '$\varphi(a)$' is same.

(b) Compute $a^{-1}ba$, where $a = (1\ 3\ 5)\ (1\ 2)$ and $b = (1\ 5\ 7\ 9)$ are elements of Sq.

(c) Is $2 + i$, prime in J[i] ? Justify your answer.

(d) Show that $11\mathbb{Z}$ is a maximal ideal of $\mathbb{Z}$.

(e) Prove with justification that the quotient ring $\mathbb{Z}_3[x] / <x^2 + 1>$ is a field, where $<x^2 + 1>$ denotes the ideal generated by $(x^2 + 1)$.

(f) Find a basis of $\mathbb{C}^2$ over $\mathbb{R}$.

(g) Let k be an extension field of F. If $a \in k$ is a root of $p(x) \in F[x]$, then show that $(x - a) / p(x)$ in $k[x]$.

(h) Which of the following is constructible ?

(i) $\sqrt[3]{2}$

(ii) $\sqrt[4]{2}$

Justify your answer.

Section – B

Answer **all** questions of the following : $16 \times 4 = 64$

3. (a) (i) If G is a group, then show that A(G), the set of automorphisms of G is also a group.

(ii) If $0(G) = p^n$, where p is a prime number, then show that $z \neq \{e\}$, where z is the center of G.

OR

(b) (i) Prove that the set of even permutations A_n in S_n form a sub group of S_n.

(ii) If G is a group, H a subgroup of G and S is the set of all right cosets of H in G, then show that there is a homomorphism θ of G into A(S), whose kernel is the largest normal subgroup of G which is contained in H.

4. (a) (i) If G is a finite group, p is a prime and p^n /o (G) but p^{n+1} X O(G), then show that any two subgroups of G of order p^n are conjugate.

(ii) Let R be a commutative ring with unit element and M be an ideal of R. If M is maximal, then show that R/M is a field. Show that the ideal generated by a prime number p is maximal in $\mathbb{Z}$.

OR

(b) (i) Prove that a Euclidean ring is a principal idea domain.

(ii) Prove that the integral domain $J[i]$ is a Euclidean ring.

5. (a) (i) Prove that $Q[x] / <x^2+1>$ is a field isomorphic to the field of complex numbers, where $<x^2+1>$ is the ideal generated by (x^2+1).

(ii) If a vector space V is of dimension n over F, then show that V is isomorphic to F^n. Also show that two vector spaces over F of same dimension are isomorphic.

OR

(b) (i) If f(x) and g(x) are primitive polynomials, then prove that f(x) g(x) is a primitive polynomial.

(ii) Let V be the set of all sequences $(a_1, a_2, \cdots\cdots, a_n, \cdots\cdots)$, $a_i \in \mathbb{R}$, where