(b) Let H be a Hilbert space, G be a subspace of H and g be a continuous linear functional on G. Then prove that there is a unique continuous linear functional f on H such that $f_{|G} = g$ and $\|f\| = \|g\|$.

5. (a) Let H be a Hilbert space and $A \in BL(H)$. Then prove that there is unique $B \in BL(H)$ such that for all $x, y \in H$,

$$\langle A(x), y \rangle = \langle x, B(y) \rangle$$

OR

(b) State and prove Generalized Schwarz Inequality.

M.A./M.Sc.-Math.-IVS-(CC-402)

2017

Time : **3** hours

Full Marks : **80**

The figures in the right hand margin indicate marks.
Answer from both the Sections as directed.

(FUNCTIONAL ANALYSIS – II)

SECTION – A

1. *Answer any **four** of the following :* *(4x4=16)*

(a) Let X be a separable normed space. Then prove that every bounded sequence in X' has a weak* convergent subsequence.

(b) Prove that ℓ^p is reflexive for $1 < p < \infty$

(c) Let $\{u_1, u_2, ...\}$ be a countable orthonormal set in an inner product space X and $x \in X$. Then prove that

$$\sum_n |\langle x, u_n \rangle|^2 \leq \|x\|^2$$

where equality holds if and only if $x = \sum_n \langle x, u_n \rangle u_n$

(d) Let H be a Hilbert space and $A \in BL(H)$. Then prove that A is normal if and only if $\|A(x)\| = \|A^*(x)\|$

(e) Let (x_n) be a sequence in a Hilbert space H. Then prove that $x_n \to x$ if and only if $x_n \xrightarrow{\omega} x$ and $\lim Sup_{n\to\infty} \|x_n\| \leq \|x\|$

(f) If H has denumerable orthonormal basis then prove that every orthonormal basis for H is denumerable.

OR

2. *Answer **all** questions :* ***(2x8=16)***

(a) State Helly's selection principle.

(b) Prove that ℓ^1 is not reflexive.

(c) Let $\{x_1, \dots \dots, x_n\}$ be an orthogonal set in X then prove that
$$\|x_1 + \cdots + x_n\|^2 = \|x_1\|^2 + \cdots + \|x_n\|^2$$

(d) Let X be an inner product space, $\{u_1, u_2, \dots\}$ be a countable orthonormal set in X and $k_1, k_2, \dots$ belong to **K.** If $\sum_n k_n u_n$ converges to some x in X, then prove that $\langle x, u_n \rangle = k_n$ for each n and $\sum_n |k_n|^2 < \infty$

(e) State polarization identity.

(f) Let X is a reflexive normed space then prove that X' is reflexive.

(g) Let X be an inner product space. Let $E \subset X$ and $x \in \bar{E}$. Then prove that there exists a best approximation from E to x if and only if $x \in E$.

(h) Let H be a Hilbert space and $A \in BL(H)$. Then prove that $Z(A) = R(A^*)^{\perp}$.

(Turn over)

SECTION – B

*Answer **all** questions* ***(16x4=64)***

3. (a) Let (z_n) be a sequence of non decreasing functions on $[a,b]$ such that $\alpha \le z_n(t) \le \beta$ for some constant α, β all $n = 1,2, \dots$ and $t \in [a,b]$. Then prove that there is a non decreasing function z on $[a,b]$ such that z is right continuous on (a,b) and for some subsequence $\left(z_{n_j}\right)$ of (z_n), we have $z_{n_j}(a) \to z_n(a)$, $z_{n_j}(b) \to z_n(b)$ and $z_{n_j}(t) \to z_n(t)$ for every $t \in (\in a, b)$ at which z is continuous.

OR

(b) State and prove Helly's theorem

4. (a) Let $X = C^1([a,b])$, the linear space of all scalar valued continuously differentiable functions on $[a,b]$. For x and y in X, define
$$\langle x, y \rangle_a = x(a)\bar{y}(a) + \int_a^b x'(t)\bar{y}'(t)\,dt$$
Prove that X is inner product space but not Hilbert Space.

OR

(b) State and prove the Gram-Schmidt Orthonormalization theorem.

5. (a) State and prove the Riesz representation theorem.

OR

(Turn over)