January, 2017

ELEMENTARY COMPLEX ANALYSIS

Time : Three Hours] [*Maximum Marks* : 80

Answer from both the Sections as directed. The figures in the right-hand margin indicate marks.

SECTION-A

1. Answer any **four** of the following : 4×4

(*a*) Determine the regions of Argand diagram defined by $|z-1| + |z+1| \leq 4$.

(*b*) Show that an equation of the form $z\bar{z} + b\bar{z} + \bar{b}z + c = 0$, *c*-real, represents a circle. Find its centre and radius.

(*c*) Construct the analytic function $f(z) = u + iv$ of which the real part is :

$$U = e^x (x \cos y - y \sin y)$$

(*d*) Prove that real and imaginary parts of an analytic function satisfy Laplace's equation.

(*e*) Evaluate $\int_c \left(z^2+3z+z\right)dz$ where c is the arc of cycloid $x = a(\theta + \sin\theta)$, $y = a(1-\cos\theta)$ between the points (0, 0) and $(\pi a, 2a)$.

(*f*) Find the bilinear transformation which maps 0, 1 and ∞ into 1, i and -1 respectively.

OR

2. Answer **all** the questions from the following : 2×8

(*a*) Fnd the modulus and argument of $\frac{1-i}{1+i}$.

(*b*) For any two complex numbers z_1 and z_2. Prove that $|z_1+z_2| \le |z_1| + |z_2|$.

(*c*) Define analytic function with an example.

(*d*) Write Cauchy-Riemann equations in polar form for an analytic function.

(*e*) Evaluate $\int_c \frac{dz}{z}$, where c is the circle with centre at origin and radius r.

(*f*) State Cauchy-Goursat theorem for complex integration of an analytic function.

(*g*) Define a conformal transformation.

(*h*) Find the fixed points and normal form of the bilinear transformation $\omega = \frac{z-1}{z+1}$.

SECTION-B

Answer **all** questions of the following : 16×4

3. (*a*) (*i*) Prove that $\left|\frac{z-1}{z+1}\right|$ = constant and $\text{amp}\left(\frac{z-1}{z+1}\right)$ = constant are orthogonal circles.

(*ii*) State and prove Cauchy's inequality for complex numbers. Derive necessary and sufficient conditions for equality.

OR

(*b*) (*i*) Prove that the area of the triangle whose vertices are the points z_1, z_2, z_3 on the Argand diagram is

$$\sum\left\{\frac{(z_2 - z_3)|z|^2}{4iz_1}\right\}.$$

Also show that the triangle is equilateral if

$$z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1$$

(*ii*) Find the equation of a straight line joining two points z_1 and z_2. Also show that the points a and a' are the inverse points with respect to the line $z\bar{b} + \bar{z}b = c$ if $a'\bar{b} + \bar{a}b = c$.

4. (*a*) (*i*) Prove that the one-valued function $f(z) = u\,(x,y) + iv\,(x,y)$ is analytic in a domain D if the four partial derivatives u_x, v_x, u_y, v_y exist, are continuous and satisfy the Cauchy-Riemann equations at each point D.

(*ii*) If $f(z) = u + iv$ is an analytic function of $z = x + iy$ and

$$u - v = \frac{e^y - \cos x + \sin x}{\cosh y - \cos x}$$

find $f(z)$ subject to the condition

$$f\left(\frac{\pi}{2}\right) = \frac{3-i}{2}.$$

OR

(*b*) Define radius of convergence of a power series. Prove that the sum function $f(z)$ of the series $\sum_{n=0}^{\infty} a_n z^n$ represents an analytic function inside its circle of convergence.

5. (*a*) State and prove Cauchy's integral formula for higher order derivatives. Using this formula, prove that

$$\int_c \frac{e^{2z}}{(z+1)^4}\,dz = \frac{8\pi e^{-2}}{3}i,$$

where c is the circle $|z| = 2$.

OR

(*b*) (*i*) Let $f(z)$ be analytic in a simply connected region D of the complex plane. Then show that there exists a function $F(z)$ analytic in D such that

$$F'(z) = f(z) \ (z \in D).$$

(*ii*) Using Cauchy's integral formula, evaluate

(1) $\int_c \frac{e^z}{(z+1)^2}dz$ where c is the circle $|z-1|=3$.

(2) $\int_c \frac{\log z}{(z-1)^3}dz$ where c is the circle $|z-1|=\frac{1}{2}$.

6. (*a*) (*i*) State and prove necessary condition for a transformation $w = f(z)$ to represent a conformal mapping.

(*ii*) Discuss the application of the transformation $w=z^2$ to the area in the first quadrant of the z-plane bounded by the axes and the circles $|z|=a$, $|z|=b$ $(a>b>0)$.

OR

(*b*) (*i*) Find the bilinear transformation which transforms half plane $R(z)\geq 0$ onto the unit circular disc $|w|\leq 1$.

(*ii*) Find the image of the infinite strips $\frac{1}{4}<y<\frac{1}{2}$ and $0<y<\frac{1}{2}$ under the transformation $w=\frac{1}{2}$. Show the region graphically.