OR

(*b*) (*i*) Prove that a polynomial domain $F[x]$ over a field F is a principal ideal domain.

(*ii*) Show by an example that the union of two subspaces need not be a subspace. Also prove that the union of two subspaces is a subspace if and only if one is contained in the other.

6\. (*a*) (*i*) Let Q be the field of rationals, then show that $Q\left(\sqrt{2},\sqrt{3}\right)=Q\left(\sqrt{2}+\sqrt{3}\right)$.

(*ii*) Define a constructible real number. If $a>0$ is constructible, then show that $\sqrt{a}$ is also constructible. Also show that if a, b are constructible numbers, then $a \pm b$, ab, $ab^{-1}\ (b \neq 0)$ are also constructible.

OR

(*b*) (*i*) Prove that a minimal splitting field of a non-constant polynomial $f(x) \in K[x]$ over K is normal extension of K.

(*ii*) If M is a finite extension field of a field K and K is finite extension field of a Field F such that $F \subset K \subset M$, then show that M is a finite extension of F and [M:F] = [M:K] [K:F]

January, 2017

ALGEBRA - I

Time : Three Hours] [*Maximum Marks* : 80

Answer from both the Sections as directed. The figures in the right-hand margin indicate marks.

SECTION-A

1\. Answer any **four** of the following : 4×4

(*a*) If f is an automorphism on a group G and H is a normal subgroup of G, then show that $f(H)$ is also a normal subgroup of G.

(*b*) Find the number of distinct cycles of length r in the permutation group S_n.

(*c*) Prove that the characteristic of an integral domain is either zero or a prime number.

(*d*) Show that a commutative ring without zero divisors can be embeded in a field.

(*e*) Prove that any two bases of a finite-dimensional vector space, have the same number of elements.

(*f*) Let $f(x)=x^4+x^2+1 \in Q[x]$. Show that the splitting field of $f(x)$ over Q is $Q(w)$, and $[Q(w) : Q]=2$.

OR

2. Answer all questions of the following : 2×8

(a) Let G be a group and $g \in G$. Then show that $T_g : G \to G$ defined by $T_g(x) = gxg^{-1}$; $\forall x \in G$, is an automorphism.

(b) Find the inverse of $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 2 \end{pmatrix}$.

(c) Define conjugate subgroups of a group.

(d) Define a maximal ideal.

(e) Show that the vectors $(0,1,-2)$, $(1,-1,1)$, $(1,2,1)$ are linearly independent in $\mathbb{R}^3(\mathbb{R})$.

(f) Define dimension of a vector space.

(g) Find the degree of splitting field of $x^4 - 5x^2 + 6$ over Q.

(h) Show that $Q\left(\sqrt{-2}\right)$ is normal extension of Q.

SECTION-B

Answer all questions of the following : 16×4

3. (a) (i) Prove that the set $I_n(G)$ of all inner automorphism of a group G is normal subgroup of Aut(G), the group of automorphism in G.

(ii) Show that every finite group G is isomorphic to a permutation group.

OR

(b) (i) If G is an infinite cyclic group, then prove that Aut (G), the group of automorphism in G is isomorphic to $\mathbf{Z}_2$.

(ii) Define conjugacy relation in a group. Find the conjugacy class of elements of permutation group S_3.

4. (a) (i) State and prove Sylow's first theorem.

(ii) Prove that every field is an integral domain, but the converse is not true.

OR

(b) (i) Show that in a group of order 30, there exists a normal subgroup of order 15.

(ii) Show that the set

$Q\left[\sqrt{-2}\right] = \left\{a + b\sqrt{2};\ a, b \in Q\right\}$ is a field with respect to addition and multiplication.

5. (a) (i) If $g(x) \neq 0$ and $f(x)$ are any two polynomials over a field F, show that there exist unique polynomials $q(x)$ and $r(x)$ over the field F such that $f(x) = g(x).q(x) + r(x)$

Where $r(x) = 0$ or $\deg(r(x)) < \deg(g(x))$.

(ii) Define linear dependent and linear independent vectors. Prove that the set of non-zero vectors $u_1, u_2 \ldots u_n$ of a vector space $V(F)$ is linear dependent if and only if some u_r, $2 \leq r \leq n$ is a linear combination of the preceding ones.