January, 2017

PARTIAL DIFFERENTIAL EQUATIONS AND APPLICATION

Time : Three Hours] [*Maximum Marks* : 80

Answer from both the Sections as directed. The figures in the right hand margin indicate marks.

SECTION-A

1. Answer any **four** of the following : 4×4

(*a*) Find the general solution of

$u_{xx} + u_x = 0$ by setting $u_x = v$

(*b*) Solve the Cauchy problem

$u_t + uu_x = 0$

$u(x, 0) = h(x)$

(*c*) Explain the Heat Conduction problem.

(*d*) Prove that the Fourier transform F is linear.

(*e*) Find the Fourier Cosine Transform of $f(x)=e^{-2x}+4e^{-3x}$.

(*f*) Find the inverse Laplace transform of

$$\frac{s^2+2}{s\left(s^2+4\right)}.$$

OR

2. Answer all questions : 2×8

(*a*) Find the Eigen Values of the equation :

$$y''+\lambda y=0;\ y(0)=y(\pi);\ y'(0)=y'(\pi)$$

(*b*) Defined Cauchy problem with example.

(*c*) State the Cauchy-Kowalewskaya Theorem.

(*d*) Explain Semi-infinite with a fixed end.

(*e*) What is the displacement of the struck string ?

(*f*) State Uniqueness Theorem.

(*g*) Find the Fourier transform of

$$f(x)=\begin{cases} e^{iwx} & \text{for } a<x<b \\ 0 & \text{otherwise} \end{cases}$$

(*h*) If $L\{f(t)\} = F(s)$, then prove that

$$L\{e^{at} f(t)\} = F(s-a).$$

SECTION-B

Answer all questions : 16×4

3. (*a*) (*i*) If $u(x, y)$ is a homogeneous function of degree n, show that u satisfy the first order equation $xu_x + yu_y = nu$.

(*ii*) Solve the Cauchy problem :

$(y+u)\ u_x + yu_y = (x-y)$ with $u = 1+x$, on $y=1$

OR

(*b*) (*i*) Determine the region in which the given equation is hyperbolic, parabolic or elliptic and transform the equation

$u_{xx} + u_{xy} - xu_{yy} = 0$ to canonical form.

(*ii*) Reduce the Tricomi equation

$u_{xx} + xu_{yy} = 0$ to the canonical form.

4. (*a*) (*i*) Determine the solution of the Goursat problem $u_{tt} = c^2 u_{xx}$,

$$u(x, t) = f(x) \text{ on } x - ct = 0$$

$$u(x, t) = g(x) \text{ on } t = t(x)$$

where $f(0) = g(0)$

(*ii*) Determine the solution of the initial boundary value problem :

$$u_{tt} = 9\, u_{xx},\ 0 < x < \infty,\ t > 0$$

$$u(x, 0) = 0, \quad 0 \le x < \infty$$

$$u_t(x, 0) = x^3,\ 0 \le x < \infty$$

$$u_x(0, t) = 0,\ t \ge 0$$

OR

(*b*) (*i*) Solve $3u_{xx} + 10u_{xy} + 3u_{yy} = 0$, $u(x, 0) = f(x)$, $u_y(x, 0) = g(x)$.

(*ii*) Prove the uniqueness theorem for the boundary-value problem involving the Laplace equation :

$$u_{xx} + u_{yy} = 0,\ 0 < x < a,\ 0 < y < b$$

$$u(x, 0) = f(x),\ u(x, b) = 0,\ 0 \le x \le a$$

$$u_x(0, y) = 0 = u_x(a, y),\ 0 \le y \le b$$

5. (*a*) (*i*) The Sturm-Liouville operator L is self-adjoint. In other words, for any $y, z \in D(L)$, we have $\langle L[y], z\rangle = \langle y, L[z]\rangle$, where $<,>$ is the inner product in $L^2([a, b])$ defined by

$$\langle f, g\rangle = \int_a^b f(x)\overline{g(x)}dx.$$

(*ii*) Find the eigen values and eigen functions of the following regular Sturm-Liouville system :

$$(1 + x)^2 y'' + 2(1 + x)y' + 3\lambda y = 0,$$
$$0 \le x \le 1;\ y(0) = 0,\ y(1) = 0$$

OR

(*b*) (*i*) Show that the compatibility condition for the Neumann problem $\nabla^2 u = f$ in D, $\frac{\partial u}{\partial n} = g$ on B is $\int_D f ds + \int_B g ds = 0$, where B is the boundary of domain D.

(*ii*) Solve the Dirichlet problem :

$\nabla^2 u = 0,\ 0 < x < 1,\ 0 < y < 1$

$u(x, 0) = x(x-1), u(x, 1) = 0,\ 0 \le x \le 1$

$u(0, y) = 0;\ u(1, y) = 0,\ 0 \le y \le 1$

6. (*a*) (*i*) Find the Fourier transform of the function :

$$f(x) = \begin{cases} 1 + \frac{x}{a}, & -a < x < 0 \\ 1 - \frac{x}{a}, & 0 < x < a \\ 0, & \text{otherwise} \end{cases}$$

(*ii*) Using Laplace transform method, solve

$y''(t) - 4y'(t) + 4y(t) = 64 \sin 2t$

$y(0) = 0,\ y'(0) = 1$

OR

(*b*) (*i*) Find the Fourier Cosine transform of $f(x) = 5e^{-2x} + 2e^{-5x}$.

(*ii*) Solve, using Laplace transform method :

$y''(t) + 9y(t) = 6 \cos 3t$

$y(0) = 2,\ y'(0) = 0$
